Introduction to Modular Forms

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Outline

1. Fundamental domain of the modular group

2. Modular functions and modular forms

3. The space of modular forms

The modular group

Definition

- Let H denote the upper half plane of $\mathbb{C}$, i.e. $H = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.
- Let $\tilde{\mathbb{C}}$ denote $\mathbb{C} \cup \{\infty\}$.
- Let $SL_2(\mathbb{R})$ denote the group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$.

Action of $SL_2(\mathbb{R})$

Given $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $z \in \tilde{\mathbb{C}}$, we define

$$gz := \frac{az + b}{cz + d}; \qquad g\infty := \frac{a}{c} = \lim_{z \to \infty} gz.$$

Correspondingly $g(-d/c) = \infty$ and if $c = 0$, $g\infty = \infty$.

Exercise 1

Check that g *is a group action on* $\tilde{\mathbb{C}}$, *i.e.* $g_1(g_2(z)) = g_1 \cdot g_2(z)$.

Exercise 2

Show that $\pm I = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *are the only elements of* $SL_2(\mathbb{R})$ *that acts trivially on* $\tilde{\mathbb{C}}$.

Thus $PSL_2(\mathbb{R}) := SL_2(\mathbb{R})/\{\pm I\}$ acts *faithfully*.

Exercise 3

Show that $\mathrm{Im}(gz) = |cz + d|^{-2}\mathrm{Im}(z)$.

Hence the action of $SL_2(\mathbb{R})$ preserves H.

Definition

Let $SL_2(\mathbb{Z})$ be the subgroup of $SL_2(\mathbb{R})$ with integer entries. Then $\Gamma := SL_2(\mathbb{Z})/\{\pm I\}$ is also called the modular group.

Fundamental domain

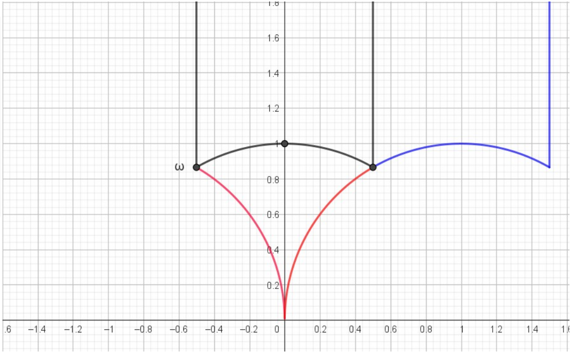
Definition

If G is a subgroup of Γ, we say two points $z_1, z_2 \in H$ are G-equivalent if there exists $g \in G$, such that $gz_1 = z_2$. A closed region F in H is called a fundamental domain for G if every $z \in H$ is G-equivalent to a point in F, but no two distinct points in the interior of F are G-equivalent.

Theorem 1

A fundamental domain for Γ is given by

$$D = \left\{ z \in H \,\middle|\, -\frac{1}{2} \leqslant \mathrm{Re}(z) \leqslant \frac{1}{2} \text{ and } |z| \geqslant 1 \right\}.$$



- What is the value of ω?
- Define $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Fundamental domain

Proof of Theorem 1: Let Γ' be the subgroup of Γ generated by S and T. We will prove that for any $z \in H, \exists g' \in \Gamma'$ such that $g'z \in D$.

Let $z \in H$ be fixed. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma'$, then $\mathrm{Im}(gz) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$. As c and d run through all possible integers, $|cz+d|$ is bounded away from 0. In other words, there are only finitely many pairs (c,d) such that $|cz+d|$ is less than a given number. So we can choose $g \in \Gamma'$ such that $\mathrm{Im}(gz)$ is maximum. We can choose a suitable integer n such that $z' = T^n(gz)$ has real part in the range $[-\frac{1}{2}, \frac{1}{2}]$, i.e. z' belongs to D. Note that we must have $|z'| \geqslant 1$, otherwise $S(z')$ would have a larger imaginary part. Thus $g' = T^n g$.

Next,we show that no two distinct points in the interior of D are Γ-equivalent. Suppose there exist $z \in D$ and $gz \in D$, and that $\mathrm{Im}(gz) \geqslant \mathrm{Im}(z)$. This would imply $|cz+d| \leqslant 1$. We have

$$1 \geqslant |cz+d| \geqslant |\mathrm{Im}(cz+d)| = |\mathrm{Im}(cz)| = |c|\mathrm{Im}(z).$$

So $c = 0$ or ± 1.

If $c = 0$, then $d = \pm 1$, i.e. $g = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$. Since $gz \in D$, either $b = 0$ (i.e. $gz = z$) or $b = \pm 1$. This means z and gz have real parts $\pm\frac{1}{2}$.

Fundamental domain

Proof of Theorem 1 cont'd: If $c = 1$, i.e. $1 \geqslant |z + d| \geqslant |\mathrm{Re}(z + d)|$. This means $d = 0$ or ± 1.

If $d = 0$, $|z| \leqslant 1$, and so we must have $|z| = 1$. As $ad - bc = 1$, we have $b = -1$, i.e. $gz = a - \frac{1}{z}$. Since $gz \in D$, either $a = 0$ (i.e. $g = S$) or $a = 1$ (i.e. $z = -\overline{\omega}$, $g = TS$), or $a = -1$ (i.e. $z = \omega$, $g = T^{-1}S$).

If $d = 1$, $\mathrm{Re}(z) = -\frac{1}{2}$, i.e. $z = \omega$, $g = \begin{pmatrix} a & a-1 \\ 1 & 1 \end{pmatrix}$, so $g\omega = a - \frac{1}{1+\omega} = a + \omega$. Thus $a = 0$ ($g = ST$) or $a = 1$ ($g = -ST^{-1}S$).
If $d = -1$, $\mathrm{Re}(z) = \frac{1}{2}$, i.e. $z = -\overline{\omega}$.
Finally, if $c = -1$, just change the sign of a, b and d. □

Exercise 4

Verify that $T^{-1}S = (ST)^2$ *and* $(ST)^3 = -I$.

Corollary 2

Γ is generated by S and T, i.e. $\Gamma = \Gamma'$.

Proof: Let $g \in \Gamma$ and choose a point z_0 in the interior of D. Let $z = gz_0$, then there exists $g' \in \Gamma'$ such that $g'z \in D$. Hence z_0 and $g'z = g'gz_0$ are Γ-equivalent which means $g'g = I$, thus $g \in \Gamma'$. $\square$

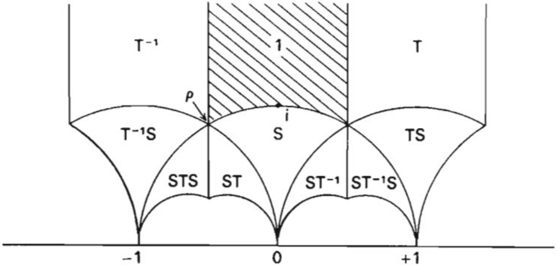


Figure: Serre [p. 78]

Remark: One can do the same calculations for subgroups of Γ, which will have “larger” fundamental domains. An example from Koblitz [p. 106] is $\Gamma(2)$ where $[\Gamma : \Gamma(2)] = 6$ with three cusps $\infty, 0, -1$.

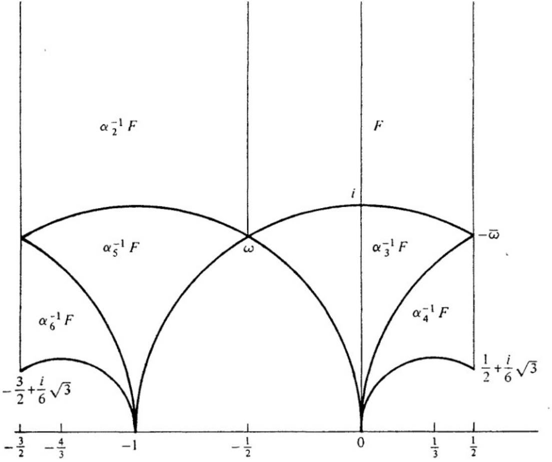


Figure III.3. A fundamental Domain $F(2)$ for $\Gamma(2)$.

Modular functions

Definition

Let k be an integer and $f(z)$ be meromorphic on H. Suppose $f(z)$ satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Since Γ is generated by S and T, it suffices to check $f\left(-1/z\right) = z^k f(z)$ and $f(z+1) = f(z)$, i.e. $f(z)$ is periodic with period 1. We can express f as

$$f(z) = \sum_{n\in\mathbb{Z}} a_n q^n, \qquad \text{where } q = e^{2\pi i z}.$$

Suppose further that f is meromorphic at infinity, i.e. at most finitely many nonzero a_n for $n<0$, then we say $f(z)$ is a modular function of weight k for Γ.

Modular forms

Definition

A modular function which is holomorphic everywhere (including infinity) is called a modular form. A modular form of weight k for Γ is thus given by

$$f(z) = \sum_{n=0}^{\infty} a_n q^n = \sum_{n=0}^{\infty} a_n e^{2\pi i n z},$$

which converges for $|q| < 1$, i.e. $\mathrm{Im}(z) > 0$ and satisfies $f(-1/z) = z^k f(z)$.

If $a_0 = 0$, then $f(z)$ is called a cusp-form.

Definition

$M_k(\Gamma)$ is defined to be the set of modular forms of weight k for Γ.
Correspondingly, $S_k(\Gamma)$ is defined to be the set of cusp-forms of weight k for Γ.

Ramanujan's τ function

Example

Ramanujan's τ function, a cusp form of weight 12, is defined by

$$\sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \cdots .$$

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$|\tau(n)| \le n^{11/2} d(n)$

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MacDonald–Dyson formula

$$\tau(n) = \sum \frac{(a-b)(a-c)(a-d)(a-e)(b-c)(b-d)(b-e)(c-d)(c-e)(d-e)}{4!3!2!1!}$$

summed over integers a, b, c, d, e, with
$a, b, c, d, e \equiv 1, 2, 3, 4, 5 \pmod 5$,
$a + b + c + d + e = 0$,
$a^2 + b^2 + c^2 + d^2 + e^2 = 10n$.

Remarks on Modular functions

If k is an odd integer, there are no nonzero modular functions of weight k for Γ.

Applying $g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ gives $f(z) = (-1)^k f(z)$.

The set of modular functions/forms of the same weight forms a complex vector space under addition and scalar multiplication. If f_1 and f_2 are modular functions/forms of weights k_1 and k_2 respectively, then the product $f_1 f_2$ is a modular function/form of weight $k_1 + k_2$.

Exercise 5

Check that if the transformation $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ *holds for* g_1 *and* g_2*, then it holds for the product* $g_1 g_2$*.*

Hence it suffices to check the transformation holds for S and T.

Eisenstein series

Definition

Let k be an even integer greater than 2, for $z \in H$, define

$$G_k(z) = \sum_{m,n}{}' \frac{1}{(mz+n)^k},$$

where the sum is over all integers m, n not both zero.

Clearly, $G_k(z+1) = G_k(z)$. Next,

$$G_k(-1/z) = \sum_{m,n}{}' \frac{1}{(-\frac{m}{z}+n)^k} = \sum_{m,n}{}' \frac{z^k}{(-m+nz)^k} = z^k G_k(z).$$

So $G_k(z)$ satisfies the transformations for T and S.

Remark: Note we have switched the order of summation in the above and this is guaranteed by absolute convergence, i.e. $k > 2$.

Eisenstein series

Theorem 3

$G_k \in M_k(\Gamma)$.

Proof: Since $k \geqslant 4$, the double sum is absolutely convergent, and uniformly convergent in any compact subset of H. Hence $G_k(z)$ is holomorphic in H. We also need to check that $G_k(z)$ is holomorphic at infinity, i.e.

$$\lim_{z \to i\infty} \sum_{m,n}{}' \frac{1}{(mz+n)^k} = \sum_{n \neq 0} \frac{1}{n^k} = 2\zeta(k),$$

where ζ denote the Riemann zeta function. □

q-expansion of Eisenstein series

Theorem 4

Let $k \geqslant 2$ *be a positive integer and* $z \in H$. *Then*

$$G_{2k}(z) = 2\zeta(2k)\left(1 - \frac{4k}{B_{2k}}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n\right),$$

where $q = e^{2\pi iz}$, $\sigma_{2k-1}(n) = \sum_{d|n} d^{2k-1}$ *and the Bernoulli numbers,* B_k, *are defined by*

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

Exercise 6

Verify that $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$ *and* $B_4 = -\frac{1}{30}$. *[Cross multiply and compare coefficients.]*

Theorem 5

Let k *be a positive integer, then* $\zeta(2k) = \sum_{n\geqslant 1} \frac{1}{n^{2k}} = (-1)^{k+1}\frac{2^{2k-1}\pi^{2k}}{(2k)!}B_{2k}$.

Proof: Recall the formula $\sin(z) = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\pi^2 n^2}\right)$. Taking the logarithmic derivative,

$$\frac{\cos(z)}{\sin(z)} = \frac{zi(e^{iz}+e^{-iz})}{e^{iz}-e^{-iz}} = \frac{1}{z} + \sum_{n\geqslant 1}\frac{1}{1-\frac{z^2}{\pi^2n^2}}\times\left(-\frac{2z}{\pi^2n^2}\right) = 1 - 2\sum_{n\geqslant 1}\frac{z^2}{\pi^2n^2-z^2}$$

$$\implies iz + \frac{2ize^{-iz}}{e^{iz}-e^{-iz}} = iz + \frac{2iz}{e^{2iz}-1} = 1 - 2\sum_{n\geqslant 1}\frac{z^2}{\pi^2n^2}\sum_{k\geqslant 0}\left(\frac{z^2}{\pi^2n^2}\right)^k$$

$$\implies iz + \sum_{k\geqslant 0}B_k\frac{(2iz)^k}{k!} = 1 - 2\sum_{n\geqslant 1}\sum_{k\geqslant 1}\left(\frac{z^2}{\pi^2n^2}\right)^k.$$

Comparing coefficients, $B_0 = 1$, $B_1 = -\frac{1}{2}$. Since right side is purely real, $B_{2j+1} = 0$, for $j \geqslant 1$, and the theorem follows from coefficients of z^{2k}. □

Exercise 7

Verify that $\zeta(2) = \frac{\pi^2}{6}$ *and* $\zeta(4) = \frac{\pi^4}{90}$.

Proof of Theorem 4: Replace z by πz and divide throughout by z in the cot formula to get

$$\pi i + \frac{2\pi i}{e^{2\pi i z} - 1} = \frac{1}{z} - \sum_{n \geqslant 1} \frac{2z}{(n-z)(n+z)} = \frac{1}{z} + \sum_{n \geqslant 1} \frac{1}{z+n} + \frac{1}{z-n}$$

$$\Longrightarrow \pi i - 2\pi i \sum_{n \geqslant 0} e^{(2\pi i z)n} = \frac{1}{z} + \sum_{n \geqslant 1} \frac{1}{z+n} + \frac{1}{z-n}.$$

Differentiating $2k-1$ times,

$$-(2\pi i)^{2k} \sum_{n \geqslant 1} n^{2k-1} e^{2\pi i n z} = (-1)^{2k-1}(2k-1)! \sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^{2k}}.$$

Therefore (with z replaced by mz),

$$G_{2k}(z) = \sum_{m,n}{}' \frac{1}{(mz+n)^{2k}} = 2\zeta(2k) + 2\sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \frac{1}{(mz+n)^{2k}} = 2\zeta(k) + \frac{2(-1)^k (2\pi)^{2k}}{(2k-1)!} \sum_{m,n \geqslant 1} n^{2k-1} e^{2\pi i m n z}.$$

Proof of Theorem 4 cont'd: Replace $e^{2\pi iz}$ with q, we have

$$\begin{aligned} G_{2k}(z) &= 2\zeta(2k) - \frac{8k}{B_{2k}} \times \frac{(-1)^{k+1}2^{2k-1}\pi^{2k}B_{2k}}{(2k)!} \sum_{m,n\geqslant 1} n^{2k-1}q^{mn} \\ &= 2\zeta(2k)\left(1 - \frac{4k}{B_{2k}}\sum_{n\geqslant 1}\sigma_{2k-1}(n)q^n\right). \quad \square \end{aligned}$$

Normalized Eisenstein series

Definition

For $k \geqslant 2$, define the normalized Eisenstein series as

$$E_{2k} = 1 - \frac{4k}{B_{2k}} \sum_{n \geqslant 1} \sigma_{2k-1}(n) q^n.$$

We have

$$E_4(z) = 1 + 240 \sum_{n \geqslant 1} \sigma_3(n) q^n; \qquad E_{10}(z) = 1 - 264 \sum_{n \geqslant 1} \sigma_9(n) q^n;$$

$$E_6(z) = 1 - 504 \sum_{n \geqslant 1} \sigma_5(n) q^n; \qquad E_{12}(z) = 1 + \frac{65520}{691} \sum_{n \geqslant 1} \sigma_{11}(n) q^n;$$

$$E_8(z) = 1 + 480 \sum_{n \geqslant 1} \sigma_7(n) q^n; \qquad E_{14}(z) = 1 - 24 \sum_{n \geqslant 1} \sigma_{13}(n) q^n.$$

Exercise 8

Verify the first four terms of the q-expansions of E_4 and E_6 are

$$E_4 = 1 + 240q + 2160q^2 + 6720q^3 + \cdots \quad \text{and} \quad E_6 = 1 - 504q - 16632q^2 - 122976q^3 + \cdots .$$

Definition

The weight 12 cusp-form $\Delta(z)$ is defined as

$$\Delta(z) = \frac{(2\pi)^{12}}{1728} \left(E_4(z)^3 - E_6(z)^2 \right) .$$

Exercise 9

Compute the first two nonzero coefficients of $\Delta/(2\pi)^{12}$, i.e. coefficients of q and q^2.

Definition

Let f be meromorphic on H, not identically zero, and let p be a point in H. The order of f at p, denoted, $v_p(f)$, is defined as the integer n such that $f/(z-p)^n$ is holomorphic and nonzero at p.

When f is a modular function on Γ, $v_p(f) = v_{gp}(f)$ for $g \in \Gamma$, so it suffices to study the zeros and poles of f in $\Gamma \backslash H$, i.e. the fundamental domain with the Γ-equivalent boundary points identified.

Theorem 6 (Valence formula)

Let f be a nonzero modular function of weight k for Γ, then

$$v_\infty(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_\omega(f) + \sum_{p \in \Gamma \backslash H, p \neq i, \omega} v_p(f) = \frac{k}{12}.$$

Proof of Theorem 6: Suppose f has no zero or pole on the boundary of D, except possibly at i, ω and $-\overline{\omega}$. By the residue theorem,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{p\in\Gamma\backslash H, p\neq i,\omega} v_p(f),$$

where γ is the contour indicated in the accompanying figure. Note that we can choose EA large enough that all the zeroes and poles lie below it. Specifically, $E = \frac{1}{2} + iT$ and $A = -\frac{1}{2} + iT$ and all zeroes and poles of f has imaginary parts less than T.

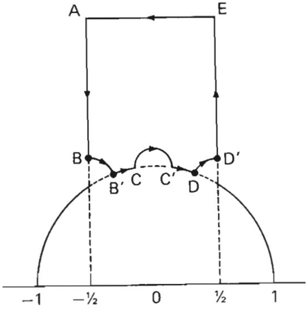

Figure: Serre, p. 86

(i) To evaluate the integral over EA, we change variables $q = e^{2\pi iz}$, so EA becomes a circle of radius $e^{-2\pi T}$ about the origin in a clockwise orientation. Hence

$$\frac{1}{2\pi i}\int_{EA} \frac{f'(z)}{f(z)}dz = -v_\infty(f).$$

Proof of Theorem 6 cont'd:

(ii) Since $f(z+1) = f(z)$, we have

$$\frac{1}{2\pi i}\int_{AB}\frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\int_{D'E}\frac{f'(z)}{f(z)}dz = 0.$$

(iii) The integral over BB' approaches the integral (clockwise) of angle $\frac{2\pi}{6}$ about ω, i.e.

$$\frac{1}{2\pi i}\int_{BB'}\frac{f'(z)}{f(z)}dz \to -\frac{1}{6}v_\omega(f).$$

(iv) Similarly, we have

$$\frac{1}{2\pi i}\int_{CC'}\frac{f'(z)}{f(z)}dz \to -\frac{1}{2}v_i(f) \quad \text{and} \quad \frac{1}{2\pi i}\int_{DD'}\frac{f'(z)}{f(z)}dz \to -\frac{1}{6}v_\omega(f).$$

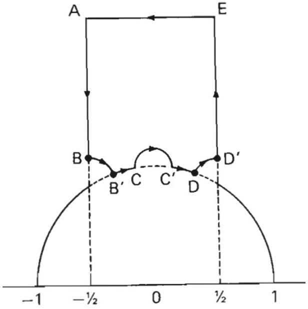

Figure: Serre, p. 86

Proof of Theorem 6 cont'd:

(v) It remains to show that

$$\frac{1}{2\pi i}\int_{B'C}\frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\int_{C'D}\frac{f'(z)}{f(z)}dz \to \frac{k}{12}.$$

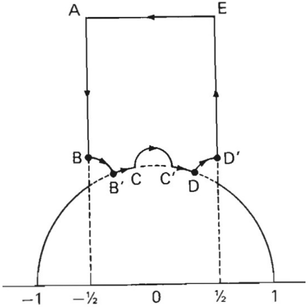

Figure: Serre, p. 86

S sends $B'C$ to DC'. (Note the orientation.) Since $f(Sz) = z^k f(z)$, we have

$$\frac{f'(Sz)}{f(Sz)}\frac{dSz}{dz} = \frac{k}{z} + \frac{f'(z)}{f(z)}.$$

$$\begin{aligned}\frac{1}{2\pi i}\int_{B'C}\frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\int_{C'D}\frac{f'(z)}{f(z)}dz &= \frac{1}{2\pi i}\int_{B'C}\frac{f'(z)}{f(z)}dz - \frac{1}{2\pi i}\int_{B'C}\frac{f'(Sz)}{f(Sz)}dSz \\ &= \frac{1}{2\pi i}\int_{B'C} -\frac{k}{z}dz \to -k \times (-\frac{1}{12}).\end{aligned}$$

If f has zeros or poles on the boundary of D, we modify the contour accordingly. □

Corollary 7

The only modular forms of weight 0 for Γ are constant functions, i.e. $M_0(\Gamma) = \mathbb{C}$.

Proof: Let $f \in M_0(\Gamma)$ and c be a value taken by $f(z)$. If $f(z)$ is holomorphic but not a constant, at least one term $v_p(f) > 0$ in Theorem 6, giving rise to a contradiction. $\square$

Corollary 8

Let k be an even integer. If $k < 0$ or $k = 2$, then $M_k(\Gamma) = 0$.

Proof: The valence formula cannot hold in either case. $\square$

Corollary 9

$M_k(\Gamma)$ is one-dimensional, generated by E_k when $k = 4, 6, 8, 10$ or 14, i.e. $M_k(\Gamma) = \mathbb{C}E_k$.

Proof: When $k = 4$, $v_\omega(f) = 1$ and there are no other zeros. Therefore f/E_4 is weight 0 modular form, i.e. a constant. $\square$

Exercise 10

Complete the proof for Corollary 9.

Corollary 10

Let k be an even integer.
(i) If $k < 12$ or $k = 14$, then $S_k(\Gamma) = 0$;
(ii) $S_{12}(\Gamma) = \mathbb{C}\Delta$;
(iii) If $k > 14$, $S_k(\Gamma) = \Delta M_{k-12}(\Gamma)$;
(iv) $M_k(\Gamma) = S_k(\Gamma) \bigoplus \mathbb{C}E_k$ for $k > 2$.

Proof: For $f \in S_k(\Gamma)$, $v_\infty(f) > 0$ and the valence formula cannot hold if $k < 12$ or $k = 14$.

When $k = 12$, $v_\infty(f) = 1$ and there are no other zeros. Thus f/Δ is a modular form of weight 0. Hence statement (ii) follows.

This also shows that the only zero of Δ is at infinity, thus if $f \in S_k(\Gamma)$ for $k > 14$, $f/\Delta \in M_{k-12}(\Gamma)$.

To prove (iv), we note that E_k does not vanish at infinity. So given $f \in M_k(\Gamma)$, we can choose an appropriate value of c such that $f - cE_k$ vanishes at infinity. Thus $f - cE_k \in S_k(\Gamma)$. $\square$

Exercise 11

Let $j \geqslant 0$ be an integer. Show that

$$\dim(M_{2j}(\Gamma)) = \begin{cases} \left\lfloor \frac{j}{6} \right\rfloor, & j \equiv 1 \pmod 6 \\ \left\lfloor \frac{j}{6} \right\rfloor + 1, & j \not\equiv 1 \pmod 6 \end{cases}.$$

Theorem 11

Any $f \in M_k(\Gamma)$ can be written in the form

$$f(z) = \sum_{4i+6j=k} c_{i,j} E_4(z)^i E_6(z)^j.$$

Proof: By induction on k. E_4, E_6, E_4^2, $E_4 E_6$ and $E_4^2 E_6$ are in $M_k(\Gamma)$ for $k = 4, 6, 8, 10$ and 14 respectively. Since the dimension of $M_k(\Gamma)$ is 1 in each case, the result holds. For $k = 12$ or $k > 14$, there exist i and j such that $4i + 6j = k$ and also $E_4^i E_6^j \in M_k(\Gamma)$.

There exists c such that $f - cE_4^i E_6^j \in S_k(\Gamma)$. In other words,

$$f = cE_4^i E_6^j + \Delta f_1 = cE_4^i E_6^j + \frac{(2\pi)^{12}}{1728} \left(E_4(z)^3 - E_6(z)^2 \right) f_1,$$

where $f_1 \in M_{k-12}(\Gamma)$. Thus f is of the required form. □

Application to divisor sums

Corollary 12

Recall that $E_4^2 = E_8$. *Hence the following divisor sum identity holds:*

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m).$$

Proof: $E_4^2 \in M_8(\Gamma)$ which is spanned by E_8. The identity holds by comparing coefficients. Thus

$$\begin{aligned} 1 + 480 \sum_{n\geqslant 1} \sigma_7(n) q^n &= \left(1 + 240 \sum_{n\geqslant 1} \sigma_3(n) q^n\right)^2 \\ &= 1 + 480 \sum_{n\geqslant 1} \sigma_3(n) q^n + 240^2 \left(\sum_{m\geqslant 1} \sigma_3(m)\right)\left(\sum_{n\geqslant 1} \sigma_3(n)\right) q^{m+n}. \end{aligned}$$

The identity follows from simplification. □

Exercise 12

Show that $E_4E_6 = E_{10}$ and $E_6E_8 = E_4E_{10} = E_{14}$. Hence obtain divisor sum identities for $\sigma_9(n)$ and $\sigma_{13}(n)$.

Remark: These divisor sum identities are classical. One can study Eisenstein series for other levels and obtain analogous results. For example, define

$$\widetilde{\sigma}_s(n) = \sum_{d|n} (-1)^{d-1} d^s,$$

$$\widehat{\sigma}_s(n) = \sum_{d|n} (-1)^{\frac{n}{d}-1} d^s,$$

then in [T., 2011]

$$48 \sum_{m<n} \widehat{\sigma}_1(m)\widehat{\sigma}_1(n-m) = -4\widehat{\sigma}_1(n) + 5\sigma_3(n) - \widetilde{\sigma}_3(n).$$

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