Introduction to Modular Forms - Exercises with Sagemath

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Exercise 1. *The matrices*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

generate the modular group $SL_2(\mathbb{Z})$. *Express the matrix*

$$\gamma = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

as a product of S *and* T.

Exercise 2. *Let* $\gamma = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$. *We compute:*

$$\det(\gamma) = 2 \cdot 2 - 3 \cdot 1 = 1 \Rightarrow \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

Apply γ *to* $z = i$*:*

$$\gamma(i) = \frac{2i+3}{i+2} = \frac{(2i+3)(2-i)}{(i+2)(2-i)} = \frac{8+i}{5}.$$

1. *Write a function* `SL2_action(gamma, z)` *that takes as input a matrix* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ *and a complex number* z *in the upper half-plane, and returns the result of the Möbius transformation:*

$$\gamma(z) = \frac{az+b}{cz+d}.$$

2. *Verify your function by checking that* `SL2_action(gamma, i)` *returns* $\frac{8}{5} + \frac{i}{5}$ *for* $\gamma = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$.
3. *Plot both* $z = i$ *and* $\gamma(i)$ *in the complex plane to visualize the transformation.*
4. *Choose some other matrices and repeat.*

Exercise 3. *Use SageMath to compute the first 40 terms in the* q*-expansions of the Eisenstein series:*

$$E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \quad E_6(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,$$

where $\sigma_k(n)$ *is the sum of the* k*-th powers of the divisors of* n. ***Tips:***

- *SageMath provides Bernoulli numbers via the built-in function* `bernoulli(k)`
- *SageMath also includes the built-in function* `sigma(n, k)` *to compute* $\sigma_k(n)$.

Modular forms and Elliptic Curves

The Modularity Theorem states that every elliptic curve $E/\mathbb{Q}$ is associated with a modular form f of weight 2 for some congruence subgroup $\Gamma_0(N)$, where N is the conductor of E. Moreover the form is cuspidal, it is an eigenform for all Hecke operators and it is "new", meaning it does not arise from any lower level (a form satisfying these requirements is a newform). The modular form has a q-expansion of the form:

$$f(q) = \sum_{n=1}^{\infty} a_n q^n,$$

and the coefficients a_p satisfy the identity:

$$a_p = p + 1 - \#E(\mathbb{F}_p),$$

for almost all primes p.
Let E be the elliptic curve defined over $\mathbb{Q}$ by the equation:

$$E : y^2 + y = x^3 - x^2 - 10x - 20.$$

1. Use SageMath to define the elliptic curve E and compute its conductor N.
2. List all newforms of weight 2 and level N using SageMath. Use the function `Newforms`
3. For each newform f, compute the first several coefficients of its q-expansion.
4. Reduce the elliptic curve E modulo several small primes p and compute $\#E(\mathbb{F}_p)$.
5. For each newform f, verify whether the identity

   $$a_p = p + 1 - \#E(\mathbb{F}_p)$$

   holds for several small primes p (e.g., $p = 3, 5, 7, 11, 13$).
6. Identify the newform corresponding to the elliptic curve E.

Repeat the same steps for another elliptic curve of your choice, for example:

$$E_1 : y^2 + xy + y = x^3 - x^2 - 2x - 1.$$

or

$$E_2 : y^2 + xy = x^3 - x^2 - 2x - 7.$$

Modular Forms and Diophantine Equations

Fermat's Last Theorem states that the Diophantine equation

$$x^n + y^n = z^n$$

has no non-zero integer solutions for $n > 2$. The proof of this theorem relies on a deep connection between elliptic curves and modular forms. Specifically, to each hypothetical solution to the Fermat equation for $n = 4$, 5, or any $n > 3$, one can associate a hypothetical elliptic curve — called a *Frey curve* — whose properties lead to contradictions when combined with the modularity theorem. Consider the Diophantine equation:

$$x^3 + y^3 = z^3$$

for which Euler knew already all possible integer solutions.

1. Use SageMath to search for integer solutions with $1 \leq |x|, |y|, |z| \leq 100$. What do you find?

2. Suppose that $x^3 + y^3 = z^3$ has a non-trivial integer solution (choose some small coprime a, b and c as above). Define the associated Frey elliptic curve by:

   $$E_{x,y,z} : y^2 = x(x - a^3)(x + b^3),$$

   where a, b are integers satisfying $a^3 + b^3 = c^3$.

   Use SageMath to:

   - Define the curve $E_{x,y,z}$ for a hypothetical solution.
   - Compute its conductor.
   - Attempt to match it to a modular form at the same level.