Introduction to Modular Forms - Exercises

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Exercise 1. *Check that g is a group action on $\widetilde{\mathbb{C}}$, i.e. $g_1(g_2(z)) = g_1 \cdot g_2(z)$.*

Exercise 2. *Show that $\pm I = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are the only elements of $SL_2(\mathbb{R})$ that acts trivially on $\tilde{\mathbb{C}}$.*

Exercise 3. *Show that $\mathrm{Im}(gz) = |cz+d|^{-2}\mathrm{Im}(z)$.*

Exercise 4. *Define $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Verify that $T^{-1}S = (ST)^2$ and $(ST)^3 = -I$.*

Exercise 5. *Check that if the transformation $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ holds for g_1 and g_2, then it holds for the product g_1g_2.*

Exercise 6. *Verify that $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$ and $B_4 = -\frac{1}{30}$. [Cross multiply and compare coefficients.]*

Exercise 7. *Verify that $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$.*

Exercise 8. *Verify the first four terms of the q-expansions of E_4 and E_6 are*

$$E_4 = 1 + 240q + 2160q^2 + 6720q^3 + \cdots \quad \textit{and} \quad E_6 = 1 - 504q - 16632q^2 - 122976q^3 + \cdots .$$

Exercise 9. *Compute the first two nonzero coefficients of $\Delta/(2\pi)^{12}$, i.e. coefficients of q and q^2.*

Exercise 10. *Give the proof of the following statement:*
$M_k(\Gamma)$ is one-dimensional, generated by E_k when $k = 4, 6, 8, 10$ or 14, i.e. $M_k(\Gamma) = \mathbb{C}E_k$.

Exercise 11. *Let $j \geq 0$ be an integer. Show that*

$$\dim(M_{2j}(\Gamma)) = \begin{cases} \left\lfloor \frac{j}{6} \right\rfloor, & j \equiv 1 \pmod 6 \\ \left\lfloor \frac{j}{6} \right\rfloor + 1, & j \not\equiv 1 \pmod 6 \end{cases}.$$

Exercise 12. *Show that $E_4E_6 = E_{10}$ and $E_6E_8 = E_4E_{10} = E_{14}$ and obtain divisor sum identities for $\sigma_9(n)$ and $\sigma_{13}(n)$.*

Exercise 13. *Use the MacDonald–Dyson formula to compute $\tau(2)$ and $\tau(3)$.*

Exercise 14. *Show that $\sigma_t(mn) = \sigma_t(m)\sigma_t(n)$ for m, n coprime integers.*

Exercise 15. *Using the first few terms of E_4 and E_6, verify numerically the identity*

$$E_4(z)^3 - E_6(z)^2 = 1728\Delta(z),$$

where Δ is the cusp form of weight 12.

Exercise 16. *Find a basis for* M_{16} *and write each basis element as a polynomial in* E_4 *and* E_6.

Exercise 17. *Recall that for any modular function* f, $a_n(f)$ *denotes the coefficient of* q^n *in the* q*-expansion of* f.

- *Show that if* $f \in M_{16}$, *satisfies* $a_1(f) = a_2(f) = 0$, *then* $f = 0$.
- *Find constants* α, β *such that* $g = \alpha E_4^4 + \beta E_4 \Delta$ *satisfies* $a_1(g) = 1$ *and* $a_2(g) = \sigma_{15}(2)$.
- *Hence calculate the constant* γ *such that* $E_{16} = 1 + \gamma \sum_{n \geq 1} \sigma_{15}(n) q^n$.

Exercise 18. *Let* $f \in M_k$. *For any* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, *we have:*

$$f'\left(\frac{az+b}{cz+d}\right) = (cz+d)^{k+2} f'(z) + \frac{k}{2\pi i} c(cz+d)^{k+1} f(z)$$

Exercise 19. *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $\tau \in H$, *and* $f \in \mathcal{O}(H)$. *Define:*

$$\gamma(z) := \frac{az+b}{cz+d}, \qquad (f|_k\gamma)(\tau) := (cz+d)^{-k} f(\gamma(z))$$

1. *Verify that* $\mathrm{SL}_2(\mathbb{Z})$ *acts on* H *from the left:*
$$\gamma'(\gamma(z)) = (\gamma'\gamma)(z), \qquad \mathrm{Id}(z) = z$$
2. *Show that the slash operator defines a right action on the space of holomorphic functions on the upper half plane* $\mathcal{O}(H)$, *i.e.:*
$$f|_k \mathrm{Id} = f, \qquad (f|_k\gamma')|_k\gamma = f|_k(\gamma' \cdot \gamma)$$
for all $f \in \mathcal{O}(H)$, $\gamma, \gamma' \in \mathrm{SL}_2(\mathbb{Z})$.

Exercise 20. *Show that for all* $n \in \mathbb{Z}_{>1}$*:*

$$(n-1)\tau(n) \equiv 0 \mod 24.$$

Hint: compute Δ' *and* $\partial_{12}\Delta$.

Exercise 21. *Let* q *be an integer, prove that* $\Gamma(q)$ *is a normal subgroup of* $\mathrm{SL}_2(\mathbb{Z})$ *and it has index in it*

$$q^3 \prod_{p|q} \left(1 - \frac{1}{p^2}\right).$$

Recall the subgroups $\Gamma_0(q)$ *and* $\Gamma_1(q)$ *from the lectures. Compute their indices in the subgroups in* $\mathrm{SL}_2(\mathbb{Z})$.

Exercise 22. *Show that* Δ *is an eigenform for all Hecke operators.*

Exercise 23. *Prove that* $\tau(n)$ *is multiplicative, i.e.,* $\tau(mn) = \tau(m)\tau(n)$ *when* $\gcd(m,n) = 1$.

Exercise 24. *Show the recursive formula for prime powers:*

$$\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1}).$$