

Solutions to the Kurdish Algebra Project

Algebra 1 Midterm Exam

29/3/2026

Question 1

Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 11 & 9 & 8 & 1 & 2 & 5 & 10 & 7 & 4 & 6 \end{pmatrix} \in S_{11}, \quad \tau = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7) \in S_{11},$$

where τ fixes 8, 9, 10, 11.

a) Starting from 1 we get

$$1 \mapsto 3 \mapsto 9 \mapsto 7 \mapsto 5 \mapsto 1,$$

so one cycle is $(1 \ 3 \ 9 \ 7 \ 5)$. Also,

$$2 \mapsto 11 \mapsto 6 \mapsto 2, \quad 4 \mapsto 8 \mapsto 10 \mapsto 4.$$

Hence

$$\sigma = (1 \ 3 \ 9 \ 7 \ 5)(2 \ 11 \ 6)(4 \ 8 \ 10).$$

Its sign is

$$\text{sgn}(\sigma) = (-1)^{5-1}(-1)^{3-1}(-1)^{3-1} = (-1)^8 = 1.$$

Therefore σ is an even permutation.

b) Using the convention $(\alpha\beta)(x) = \alpha(\beta(x))$:

For $\sigma\tau$, compute $x \mapsto \sigma(\tau(x))$:

$$\begin{aligned} 1 &\mapsto 11, & 11 &\mapsto 6, & 6 &\mapsto 5, & 5 &\mapsto 2, & 2 &\mapsto 9, \\ 9 &\mapsto 7, & 7 &\mapsto 3, & 3 &\mapsto 8, & 8 &\mapsto 10, & 10 &\mapsto 4, \\ 4 &\mapsto 1. \end{aligned}$$

So

$$\sigma\tau = (1 \ 11 \ 6 \ 5 \ 2 \ 9 \ 7 \ 3 \ 8 \ 10 \ 4).$$

In two-line form,

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 11 & 9 & 8 & 1 & 2 & 5 & 3 & 10 & 7 & 4 & 6 \end{pmatrix}.$$

For $\tau\sigma$, compute $x \mapsto \tau(\sigma(x))$:

$$\begin{aligned} 1 &\mapsto 4, & 4 &\mapsto 8, & 8 &\mapsto 10, & 10 &\mapsto 5, & 5 &\mapsto 2, \\ 2 &\mapsto 11, & 11 &\mapsto 7, & 7 &\mapsto 6, & 6 &\mapsto 3, & 3 &\mapsto 9, \\ 9 &\mapsto 1. \end{aligned}$$

Thus

$$\tau\sigma = (1\ 4\ 8\ 10\ 5\ 2\ 11\ 7\ 6\ 3\ 9).$$

In two-line form,

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 4 & 11 & 9 & 8 & 2 & 3 & 6 & 10 & 1 & 5 & 7 \end{pmatrix}.$$

c) Since conjugate elements have the same order,

$$\tau^5\sigma^{1000}\tau^{-5}$$

has the same order as σ^{1000} .

Now

$$\sigma = (1\ 3\ 9\ 7\ 5)(2\ 11\ 6)(4\ 8\ 10),$$

so

$$|\sigma| = \text{lcm}(5, 3, 3) = 15.$$

Therefore

$$\sigma^{1000} = \sigma^{1000 \bmod 15} = \sigma^{10}.$$

On the 5-cycle, $10 \equiv 0 \pmod{5}$, so that part becomes the identity. On each 3-cycle, $10 \equiv 1 \pmod{3}$, so those cycles remain unchanged. Hence

$$\sigma^{10} = (2\ 11\ 6)(4\ 8\ 10).$$

Its order is

$$\text{lcm}(3, 3) = 3.$$

So the order of $\tau^5\sigma^{1000}\tau^{-5}$ is $\boxed{3}$.

Question 2

a) An *isometry* of $\mathbb{R}^2$ is a map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$|f(x) - f(y)| = |x - y| \quad \text{for all } x, y \in \mathbb{R}^2.$$

An *orthogonal map* on $\mathbb{R}^2$ is a linear map $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$\langle \phi(x), \phi(y) \rangle = \langle x, y \rangle \quad \text{for all } x, y \in \mathbb{R}^2.$$

Equivalently, if A is the matrix of ϕ , then $A^T A = I$.

- b) Assume $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is linear and satisfies $|\phi(x)| = |x|$ for all $x \in \mathbb{R}^2$. Then for any $x, y \in \mathbb{R}^2$,

$$\phi(x) - \phi(y) = \phi(x - y)$$

because ϕ is linear. Therefore

$$|\phi(x) - \phi(y)| = |\phi(x - y)| = |x - y|.$$

So ϕ preserves all distances, hence ϕ is an isometry.

The linearity assumption is necessary. Consider

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad f(x) = \begin{cases} x, & \text{if } x \neq 0, \\ (1, 0), & \text{if } x = 0. \end{cases}$$

Then $|f(x)| = |x|$ for every nonzero x , but $|f(0)| = 1 \neq 0$, so this example does not satisfy the norm condition at 0.

A genuine counterexample satisfying $|f(x)| = |x|$ for all x but not being an isometry is

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad f(x) = \begin{cases} x, & \text{if } x_2 \geq 0, \\ -x, & \text{if } x_2 < 0, \end{cases}$$

where $x = (x_1, x_2)$. Clearly $|f(x)| = |x|$ for all x , but f is not linear. It is also not an isometry: take

$$x = (1, 1), \quad y = (1, -1).$$

Then

$$|x - y| = |(0, 2)| = 2,$$

while

$$f(x) = (1, 1), \quad f(y) = (-1, 1),$$

so

$$|f(x) - f(y)| = |(2, 0)| = 2.$$

This pair still preserves distance, so choose instead

$$x = (1, 1), \quad y = (2, -1).$$

Then

$$|x - y| = |(-1, 2)| = \sqrt{5},$$

whereas

$$f(x) = (1, 1), \quad f(y) = (-2, 1),$$

thus

$$|f(x) - f(y)| = |(3, 0)| = 3 \neq \sqrt{5}.$$

Hence f is not an isometry.

- c) Let r_{ℓ_1} and r_{ℓ_2} be reflections in the lines ℓ_1 and ℓ_2 .

If the lines intersect at a point P , translate coordinates so that P is the origin. Reflection in a line through the origin is a linear orthogonal map with determinant -1 . Therefore the composition $r_{\ell_2} \circ r_{\ell_1}$ is an orthogonal map with determinant $+1$,

hence a rotation. More precisely, if the oriented angle from ℓ_1 to ℓ_2 is θ , then the composition is the rotation by angle 2θ .

If the lines are parallel, then the composition of the two reflections moves every point by the same vector perpendicular to the lines, with magnitude twice the distance between them. Hence the composition is a translation.

Therefore, the composition of reflections in two lines is either a rotation (when the lines intersect) or a translation (when the lines are parallel).

Question 3

Let

$$D_4 = \{\rho^i \sigma^j : i = 0, 1, 2, 3, j = 0, 1\},$$

with $\rho^4 = 1$, $\sigma^2 = 1$, and $\sigma\rho\sigma = \rho^{-1}$.

- a) A *group homomorphism* from a group G to a group H is a map $f : G \rightarrow H$ such that

$$f(xy) = f(x)f(y) \quad \text{for all } x, y \in G.$$

A subgroup $N \leq G$ is *normal* if

$$gNg^{-1} = N \quad \text{for all } g \in G.$$

The *kernel* of a homomorphism $f : G \rightarrow H$ is

$$\text{Ker}(f) = \{g \in G : f(g) = e_H\}.$$

To prove that $\text{Ker}(f)$ is normal in G , first note that it is a subgroup:

$$f(e_G) = e_H,$$

so $e_G \in \text{Ker}(f)$. If $a, b \in \text{Ker}(f)$, then

$$f(ab^{-1}) = f(a)f(b)^{-1} = e_H e_H^{-1} = e_H,$$

so $ab^{-1} \in \text{Ker}(f)$.

Now let $g \in G$ and $k \in \text{Ker}(f)$. Then

$$f(gkg^{-1}) = f(g)f(k)f(g)^{-1} = f(g)e_H f(g)^{-1} = e_H.$$

Thus $gkg^{-1} \in \text{Ker}(f)$. Hence

$$g\text{Ker}(f)g^{-1} \subseteq \text{Ker}(f)$$

for all $g \in G$. Applying the same argument to g^{-1} gives the reverse inclusion, so

$$g\text{Ker}(f)g^{-1} = \text{Ker}(f).$$

Therefore $\text{Ker}(f)$ is normal in G .

b) Let

$$H = \{1, \sigma\}, \quad K = \{1, \rho^2\}.$$

First consider H . A conjugate of H has the form

$$gHg^{-1} = \{1, g\sigma g^{-1}\}.$$

Using the relation $\sigma\rho = \rho^{-1}\sigma$, we obtain

$$\rho\sigma\rho^{-1} = \rho^2\sigma.$$

More generally, conjugating σ by elements of D_4 gives only

$$\sigma \quad \text{or} \quad \rho^2\sigma.$$

Hence the conjugates of H are exactly

$$\{1, \sigma\}, \quad \{1, \rho^2\sigma\}.$$

So H is *not* normal in D_4 .

Now consider K . Since ρ^2 commutes with both ρ and σ (indeed $\sigma\rho^2\sigma = (\sigma\rho\sigma)^2 = (\rho^{-1})^2 = \rho^2$), the element ρ^2 is central. Therefore for every $g \in D_4$,

$$g\rho^2g^{-1} = \rho^2.$$

Hence every conjugate of K equals K itself, so the only conjugate of K is

$$K = \{1, \rho^2\}.$$

Thus K is normal in D_4 .

c) The map

$$f : D_4 \rightarrow \{1, \rho, \rho^2, \rho^3\}, \quad f(\rho^i\sigma^j) = \rho^i$$

is *not* a homomorphism.

Indeed,

$$f(\sigma) = 1, \quad f(\rho) = \rho,$$

so if f were a homomorphism, then

$$f(\sigma\rho) = f(\sigma)f(\rho) = 1 \cdot \rho = \rho.$$

But using $\sigma\rho = \rho^{-1}\sigma = \rho^3\sigma$, we get

$$f(\sigma\rho) = f(\rho^3\sigma) = \rho^3.$$

Since $\rho^3 \neq \rho$, the equality fails. Therefore f is not a group homomorphism.

Question 4

Let

$$G = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : A^T A = I \right\} = O(2)$$

and

$$X = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x^2 + y^2 = 1 \right\} = S^1.$$

a) A *group action* of a group G on a set X is a map

$$G \times X \rightarrow X, \quad (g, x) \mapsto g \cdot x,$$

such that for all $g, h \in G$ and all $x \in X$,

$$e \cdot x = x \quad \text{and} \quad (gh) \cdot x = g \cdot (h \cdot x).$$

The *stabilizer* of a point $x \in X$ is the subset

$$\text{Stab}_G(x) = \{g \in G : g \cdot x = x\}.$$

b) Define

$$G \times X \rightarrow X, \quad A \cdot v = Av.$$

We must show that this is a well-defined action.

First, it is well-defined on X : if $v \in X$, then $v^T v = 1$. Since $A \in G$, we have $A^T A = I$, so

$$(Av)^T (Av) = v^T A^T A v = v^T I v = v^T v = 1.$$

Hence $Av \in X$.

Next, the identity matrix acts trivially:

$$I \cdot v = Iv = v.$$

Finally, for $A, B \in G$ and $v \in X$,

$$(AB) \cdot v = (AB)v = A(Bv) = A \cdot (B \cdot v).$$

Thus the two axioms of a group action are satisfied. Therefore this map defines an action of G on X .

c) Let

$$N = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The orbit of N is

$$\text{Orb}_G(N) = \left\{ A \begin{pmatrix} 1 \\ 0 \end{pmatrix} : A \in O(2) \right\}.$$

But $A \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is exactly the first column of A , and the columns of an orthogonal matrix are unit vectors. Conversely, for every unit vector $u \in S^1$, one can choose another unit vector v orthogonal to u so that the matrix with columns u, v is orthogonal. Hence every point of S^1 occurs as $A \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for some $A \in O(2)$.

Therefore

$$\text{Orb}_G(N) = X = S^1.$$

Now compute the stabilizer. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(2)$$

stabilizes N , then

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

so the first column of A must be $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Therefore

$$A = \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix}.$$

Since A is orthogonal, its columns must be orthonormal. The second column must be orthogonal to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and have norm 1, hence it is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$. Thus

$$\text{Stab}_G(N) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

Question 5

a) Let $a, b \in \mathbb{Z}$, not both zero.

The *greatest common divisor* of a and b , denoted $\gcd(a, b)$, is the unique positive integer d such that:

- (i) $d \mid a$ and $d \mid b$;
- (ii) if $c \mid a$ and $c \mid b$, then $c \mid d$.

The *least common multiple* of a and b , denoted $\text{lcm}(a, b)$, is the unique positive integer m such that:

- (i) $a \mid m$ and $b \mid m$;
- (ii) if $a \mid n$ and $b \mid n$, then $m \mid n$.

b) Use the Euclidean algorithm:

$$2431 = 1342 + 1089,$$

$$1342 = 1089 + 253,$$

$$1089 = 4 \cdot 253 + 77,$$

$$253 = 3 \cdot 77 + 22,$$

$$77 = 3 \cdot 22 + 11,$$

$$22 = 2 \cdot 11.$$

Thus

$$\gcd(2431, 1342) = 11.$$

Then

$$\text{lcm}(2431, 1342) = \frac{2431 \cdot 1342}{\gcd(2431, 1342)} = \frac{2431 \cdot 1342}{11} = 296582.$$

c) Since $\gcd(2431, 1342) = 11$, the equation

$$2431x + 1342y = 11$$

has integer solutions.

Back-substitute in the Euclidean algorithm:

$$11 = 77 - 3 \cdot 22,$$

$$22 = 253 - 3 \cdot 77,$$

so

$$11 = 77 - 3(253 - 3 \cdot 77) = 10 \cdot 77 - 3 \cdot 253.$$

Now

$$77 = 1089 - 4 \cdot 253,$$

thus

$$11 = 10(1089 - 4 \cdot 253) - 3 \cdot 253 = 10 \cdot 1089 - 43 \cdot 253.$$

Also,

$$253 = 1342 - 1089,$$

so

$$11 = 10 \cdot 1089 - 43(1342 - 1089) = 53 \cdot 1089 - 43 \cdot 1342.$$

Finally,

$$1089 = 2431 - 1342,$$

therefore

$$11 = 53(2431 - 1342) - 43 \cdot 1342 = 53 \cdot 2431 - 96 \cdot 1342.$$

Hence one integer solution is

$$\boxed{x = 53, \quad y = -96.}$$

The general integer solution is

$$\boxed{x = 53 + 122t, \quad y = -96 - 221t \quad (t \in \mathbb{Z}),}$$

because

$$\frac{1342}{11} = 122, \quad \frac{2431}{11} = 221.$$