

Question 1. (20 points)

- a) (4 points) Define $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 3 & 11 & 9 & 8 & 1 & 2 & 5 & 10 & 7 & 4 & 6 \end{pmatrix} \in S_{11}$.

Write σ as a product of disjoint cycles, and determine its sign.

- b) (8 points) Compute $\sigma\tau$ and $\tau\sigma$, for $\tau = (1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7) \in S_{11}$.

- c) (8 points) Determine the order of $\tau^5\sigma^{1000}\tau^{-5} \in S_{11}$.
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Question 2. (18 points)

- a) (4 points) State the definition of an **isometry** and an **orthogonal map** on $\mathbf{R}^2$.
- b) (8 points) Prove that if $\phi : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ is a linear map satisfying $|\phi(x)| = |x|$, then ϕ is an isometry. Provide a counter-example showing that the condition of linearity is necessary in the previous statement.
- c) (6 points) Prove that the composition of the reflections in two lines is either a rotation or a translation.
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Question 3. (22 points) The Dihedral group $D_4 = \{\rho^i\sigma^j, i = 0, 1, 2, 3, j = 0, 1\}$ is generated by an element ρ of order 4 and an element σ of order 2 that satisfy $\sigma\rho\sigma = \rho^{-1}$.

- a) (8 points) State the definition of a **group homomorphism**, a **normal subgroup**, and the **kernel** of a homomorphism. Prove that the kernel of any homomorphism is a normal subgroup of the domain.
- b) (8 points) Let $H = \{1, \sigma\}$, and $K = \{1, \rho^2\}$. Determine all conjugates of each of H and K in D_4 and decide whether either H or K is normal in D_4 .
- c) (6 points) Consider the map $f : D_4 \rightarrow \{1, \rho, \rho^2, \rho^3\}$ defined by $f(\rho^i\sigma^j) = \rho^i$. Determine whether f is a group homomorphism.
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Question 4. (22 points)

- a) (8 points) State the definitions of a **group action** of a group G on a set X , and of the **stabilizer** of a point $x \in X$ under the action of G .
- b) (8 points) Consider the group

$$G = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : A^t \cdot A = \text{id} \right\}$$

and the set

$$X = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x^2 + y^2 = 1 \right\}.$$

Prove that the map

$$G \times X \rightarrow X, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

defines an action of G on S .

- c) (6 points) Compute the orbit and the stabilizer of the point

$$N = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Question 5. (18 points)

- a) (6 points) Given two integers a and b , state the definition of their **greatest common divisor** $\text{GCD}(a, b)$ and their **least common multiple** $\text{LCM}(a, b)$.
- b) (6 points) Compute $\text{GCD}(2\,431, 1\,342)$ and $\text{LCM}(2\,431, 1\,342)$.
- c) (6 points) Find an integer solution of the equation $2\,431x + 1\,342y = 11$, if it exists.
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