## Exercise sheet on complex tori

(1) Let $\Lambda \subset \mathbb{C}$ be a lattice, and let $\wp$ be the Weierstrass function associated to $\Lambda$; let $\wp'$ be the first derivative of $\wp$. Prove that:
 (a) $\wp$ is an elliptic function with respect to $\Lambda$;
 (b) $\wp$ is an even function;
 (c) $\wp'$ is an elliptic function with respect to $\Lambda$;
 (d) $\wp'$ is an odd function.

(2) Find the zeros of $\wp'$ on $T = \mathbb{C}/\Lambda$, with $\Lambda = \langle \omega_1, \omega_2 \rangle$.

(3) Let $T_1, T_2$ be two complex tori. Show that for any $\alpha \in \mathbb{C}$, the map $\psi_\alpha \colon T_1 \to T_2$ , $[z] \mapsto [\alpha z]$ is an isogeny.

(4) Let $\psi \colon T_1 \to T_2$ be an isogeny of complex tori, with $T_i = \mathbb{C}/\Lambda_i$, for $i = 1, 2$.
 (a) Show that there is a lift $\phi \colon \mathbb{C} \to \mathbb{C}$ of $\psi$.
 (b) Show that $\phi$ is unique up to an additive constant $c \in \Lambda_2$.
 (c) Show that there exists an $\alpha \in \mathbb{C}$ such that $\phi \colon z \mapsto \alpha z$.
 (d) Show that $\alpha\Lambda_1 \subseteq \Lambda_2$.
 (e) Conclude that there is a bijection between the set of isogenies $T_1 \to T_2$ and the set of $\alpha \in \mathbb{C}$ such that $\alpha\Lambda_1 \subseteq \Lambda_2$.

(5) Let $\phi \colon T_1 \to T_2$ be a non-trivial isogeny of complex tori of degree $k$, and let $\hat{\phi} \colon T_2 \to T_1$ be its dual. Prove that the composition of $\phi$ and $\hat{\phi}$ (in any order) is the multiplication by $k$.

(6) Show that $\mathrm{End}(\mathbb{C}/\mathbb{Z}[i]) \cong \mathbb{Z}[i]$.

(7) Let $\zeta \in \mathbb{C}$ be a third root of unity. Show that $\mathrm{End}(\mathbb{C}/\mathbb{Z}[\zeta]) \cong \mathbb{Z}[\zeta]$.