Diophantine problems and CM solutions

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The j-invariants of elliptic curves with complex multiplications, also called singular moduli, are always algebraic integers. Can they be algebraic units? This is a question posed by David Masser, which was partially solved by Habegger in 2015, and then completely by Bilu-Habegger-Kühne in 2020. Generalizations have emerged, for example by Li in 2021 concerning the values of modular polynomials. Another example is by Campagna for the case of S-units, where S is the set of prime numbers congruent to 1 modulo 3. In collaborative work with Anglés, Armana, and Bosser, we show an analogous first answer for Drinfeld modules of rank 2: for each integer q that is a power of a prime number p, there are at most a finite number of Drinfeld modules of rank 2 with CM and whose j-invariant is a unit.

As algebraic integers, singular moduli may happen to be the coordinates of some algebraic points on curves, which leads to the study of statements à la André-Oort. These also have counterparts in the Drinfeld setting. To illustrate this, we will show how to rule out the existence of Drinfeld singular moduli on a family of hyperbolas.

We will present our results, the main ideas of their proofs, and the links with questions of effectiveness in André-Oort statements in the classical case and in the Drinfeld case. If time permits, we will mention recent work in progress with Baker and Perez Pina on the existence of Drinfeld singular moduli that may be S-units.