Nepal Algebra Project 2019 Midterm exam

Tribhuvan University

July 1^{st} 2019

1. (a) Find the minimal polynomial of $\alpha = 5 - 2\sqrt{3}$ over $\mathbb{Q}$, and *prove* that it is the minimal polynomial.

(5 marks)

(b) Prove that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{3})$ and that it is a normal extension of $\mathbb{Q}$.

(5 marks)

2. Let L/K is a finite extension of degree n and let F be an intermediate field (i.e. $K \subseteq F \subseteq L$). Prove that the degree $[F : K]$ is a divisor of n. Deduce which are the intermediate fields of an extension of degree 3.

(10 marks)

3. Let $f(x) = x^3 - 4x + 1 \in \mathbb{Q}[x]$.

(a) Prove that $f(x)$ is irreducible.

(2 marks)

(b) Suppose that α is a root of $x^3 - 4x + 1$ in $\mathbb{C}$. Express α^{-1} and $(1+\alpha)^{-1}$ as linear combinations, with rational coefficients, of 1, α and α^2.

(3 marks)

(c) Prove that α^3, α^4 and α^5 are linearly independent over $\mathbb{Q}$.

(3 marks)

(d) Prove that for every integer $n \neq 0$, we have $\mathbb{Q}(\alpha^n) = \mathbb{Q}(\alpha)$.

(2 marks)

4. Let $\zeta = \sqrt{3} - \sqrt{2}$.

(a) Show that $\mathbb{Q}(\sqrt{6}) \subset \mathbb{Q}(\zeta)$.

(2 marks)

(b) Show that $\mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\zeta)$ and that $\mathbb{Q}(\sqrt{3}) \subset \mathbb{Q}(\zeta)$.

(2 marks)

(c) Determine the minimal polynomial of ζ over $\mathbb{Q}$.

(2 marks)

(d) Calculate $[\mathbb{Q}(\zeta) : \mathbb{Q}]$.

(2 marks)

(e) Prove that $\mathbb{Q}(\zeta)$ is a normal extension of $\mathbb{Q}$.

(2 marks)

5. Let $f = X^4 - 2$.

(a) Prove that $E = \mathbb{Q}(\sqrt[4]{2}, i)$ is a splitting field for f over $\mathbb{Q}$.

(2 marks)

(b) Calculate $[E : \mathbb{Q}]$ and decide whether or not the extension $E/\mathbb{Q}(i)$ is normal.

(4 marks)

(c) Write some of the intermediate subfields for the extension $E/\mathbb{Q}$.

(4 marks)