NAP 2019 - MODULE V - CLASS #7
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In this lecture we always assume $char(K) \neq 2, 3$.

- We studied all the possible transitive subgroups of S_4. Firstly we proved that such a group G must have order divisible by 4. Then we proved that the possible transitive groups are
  - S_4;
  - A_4;
  - Dihedral groups D_4 conjugated to $\langle (1234), (13) \rangle$;
  - Cyclic groups of order 4 conjugated to $\langle (1234) \rangle$;
  - The Vierergroup $N = \{(1), (12)(34), (13)(24), (14)(23)\}$.

- We proved that the Vierergroup N is normal in S_4 and that $S_4/N \simeq S_3$.
- Let $f(X)$ be an irreducible polynomial of degree 4; by a change of variables we can suppose $f(X)$ of the form
$$f(X) = X^4 + pX^2 + qX + r.$$
Then
$$\Delta = 16p^4r - 4p^3q^2 - 128p^2r^2 + 144pq^2r - 27q^4 + 256r^3.$$
Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots of $f(X)$ in a splitting field L over K, and let $G = \mathrm{Gal}(L/K)$. Then G is a transitive subgroup of S_4. Put $H = G \cap N$, and $M = L^H$. If we define
$$\beta = \alpha_1 + \alpha_2, \quad \gamma = \alpha_1 + \alpha_3, \quad \mu = \alpha_2 + \alpha_3,$$
then we have
$$L = K(\beta, \gamma, \mu), \qquad M = K(\beta^2, \gamma^2, \mu^2).$$
Moreover β^2, γ^2, μ^2 result to be the roots of the following cubic polynomial, called the *cubic resolvent* of $f(X)$:
$$g(X) = X^3 + 2pX^2 + (p^2 - 4r)X - q^2.$$
The discriminant of g coincides to that of f.

- The above analysis proves that $f(X)$ is solvable by radicals: indeed the cubic resultant is solvable by radicals in the field $M(\omega)$ (ω primitive 3rd root of unity), as we proved in last class; then we can obtain $L(\omega)$ by adjoining to $M(\omega)$ the square roots of β^2, γ^2, μ^2.