NAP 2019 - MODULE V - CLASS #6
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In this lecture we always assume $char(K) \neq 2, 3$.

- We reduced the problem of solving an irreducible cubic polynomial $f(X)$ by radicals to the problem of solving it over $K(\sqrt{\Delta})$.

- We solved this problem firstly in the case where the splitting field L/K contains a primitive 3rd root of unity ω (this is equivalent to say that $\omega \in K(\sqrt{\Delta})$). In fact, let $\alpha \in L$ be a root of $f(X)$, and σ be a generator of $\mathrm{Gal}(L/K(\sqrt{\Delta}))$. We showed that in this case $L = K(\sqrt{\Delta}, \beta)$, where

$$\beta = \alpha + \omega\sigma(\alpha) + \omega^2\sigma^2(\alpha).$$

  We proved that $\sigma(\beta^3) = \beta^3$, so that $\beta^3 \in K(\sqrt{\Delta})$. Then L is obtained by $K(\sqrt{\Delta})$ by adding a radical, and we are done.
- We showed that we can always reduce to the above case, by firstly adding a primitive 3rd root of unity ω to $K(\sqrt{\Delta})$, (it is a quadratic extension) and then solving $f(X)$ over $K(\sqrt{\Delta})$. We obtain in this way that $L(\omega)/K$ is an extension by radicals by the tower

$$L(\omega) \supseteq K(\sqrt{\Delta}, \omega) \supseteq K(\sqrt{\Delta}) \supseteq K.$$

- Then we considered the problem of finding esplicit formulas espressing the roots of $f(X)$ by radicals (*Cardano formulas*). We defined

$$\gamma = \alpha + \omega^2\sigma(\alpha) + \omega\sigma^2(\alpha)$$

  (it is a conjugate of β in $L(\omega)$), and noticed that α^3, β^3 are roots of the *quadratic resolvent*:

$$X^2 + 27qX - 27p^3.$$

This allows to compute β and γ as radicals over $K(\sqrt{\Delta}, \omega)$ and $\alpha, \sigma(\alpha), \sigma^2(\alpha)$ as $K(\omega)$-linear combinations of β and γ; namely

$$\alpha = \frac{1}{3}(\beta + \gamma)$$
$$\sigma(\alpha) = \frac{1}{3}(\omega^2\beta + \omega\gamma)$$
$$\sigma^2(\alpha) = \frac{1}{3}(\omega\beta + \omega^2\gamma)$$

- We applied the above procedure and found the roots of the rational cubic polynomial:

$$f(X) = X^3 + 3X^2 + 2X - 1.$$