NAP 2019 - MODULE V - CLASS #3
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- **Theorem:** Let L/K be a finite extension of finite fields of characteristic p, with $|K| = p^t$. Then it is Galois and $\mathrm{Gal}(L/K) = \langle \Phi^t \rangle$.

- **Classification theorem of finite abelian groups:** every finite abelian group is isomorphic to a product of cyclic groups:

$$\mathbb{Z}_{d_1} \times \ldots \times \mathbb{Z}_{d_s},$$

  where $d_1, \ldots, d_s$ can be chosen such that $d_1 | d_2 | \cdots | d_s$.

- **Corollary:** A finite abelian group is isomorphic to a product of cyclic groups of prime power order.

- **Definition:** If G is a finite group then the *exponent* of G, denoted by $exp(G)$, is the least positive integer k such that $g^k = 1, \forall g \in G$.

$$exp(G) = \mathrm{lcm}\{o(g) \ | g \in G\}.$$

- **Examples:** $exp(S_3) = 6$, $exp(S_6) = 60$, $exp(\mathbb{Z}_2 \times \mathbb{Z}_2 = 2)$, $exp(\mathbb{Z}_2 \times \mathbb{Z}_3) = 6$.

- **Corollary:** Let G be a finite abelian group. There exists an element $g \in G$ uch that $o(g) = exp(G)$.

- **Corollary:** Let K be a field and G be a finite subgroup of $K^{\times}$. Then G is cyclic.
  This corollary generalizes to an arbitrary field a well-known propery of $\mathbb{C}^{\times}$.
  In particular the multiplicative group $\mathbb{F}_{p^n}^{\times}$ of any finite field is cyclic.

- **Corollary:** Let L/K be a finite extension of finite fields. Then it is simple, that is there is an element $\alpha \in L$ such that $L = K(\alpha)$.

- **Remark:** in the notation above, if $L^{\times} = \langle \alpha \rangle$ then $L = K(\alpha)$. The converse is not true.