• Let $\mathbf{K}$ be a field. The *formal derivative* f' of a polynomial $f = \sum_n a_n x^n \in \mathbf{K}[x]$ is by definition

$$f' := \sum n a_n x^{n-1}.$$

It has all the properties of the usual derivation of functions:
$(\alpha f + \beta g)' = \alpha f' + \beta g'$, for all $f, g \in K[x]$ and $\alpha, \beta \in K$;
$(fg)' = f'g + fg'$, for all $f, g \in K[x]$.

• Let $\mathbf{K}$ be a field. Let f be a polynomial in $\mathbf{K}[x]$ and let α be a zero of f in some extension $\mathbf{L}$ of $\mathbf{K}$. Then α is a double zero of f if and only if $f'(\alpha) = 0$.

• **The separability issue**.
Let K be a field of characteristic 0. Examples of such fields are $K = \mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{Q}(x)$, $\mathbf{Q}[x]/(f)$, where $f \in \mathbf{Q}[x]$ is an irreducible polynomial, etc...
• *Let* $\mathbf{K}$ *be a field, with* $char(\mathbf{K}) = 0$*; let* f *be an irreducible polynomial in* $\mathbf{K}[x]$*. Then* f *is necessarily separable. As a consequence, every extension* $\mathbf{K} \subset \mathbf{L}$ *is necessarily separable.*

Let $\mathbf{K}$ be a field of characteristic $p > 0$.
Then $\mathbf{K}$ admits a distinguished endomorphism, the *Frobenius endomorphism*,

$$\phi \colon \mathbf{K} \to \mathbf{K}, \quad x \mapsto \phi(x) = x^p.$$

ϕ is necessarily injective, but possibly not surjective.
If ϕ is surjective, the field is called *perfect*.

Every finite field is perfect.

Every algebraic extension $\mathbf{L}$ of a perfect field $\mathbf{K}$ is separable.

On the fundamental theorem of Galois theory.
• Let $\mathbf{K} \subset \mathbf{L}$ be a field extension. Let $G := Aut_{\mathbf{K}}(\mathbf{L})$ be the Galois group. Then

$$\#G \leqslant [\mathbf{L} : \mathbf{K}].$$