• **The Fundamental Theorem of Galois Theory.**
(cf. Garling, Thm. 11.8, page 97).
Further discussion of the statement of the fundamental theorem of Galois theory.

• More about example (2) of Lecture 1:

$$\mathbf{Q} \subset \mathbf{Q}(\sqrt[3]{2}, \omega),$$

where $\omega = e^{2\pi i/3} = \frac{1}{2}(-1 + \sqrt{-3})$ is a primitive cube root of 1.
Fix a bijection between the sets $\{\sqrt[3]{2}, \sqrt[3]{2}\omega, \sqrt[3]{2}\omega^2\}$ and $\{1, 2, 3\}$, for example

$$\sqrt[3]{2} \mapsto 1, \quad \sqrt[3]{2}\omega \mapsto 2, \quad \sqrt[3]{2}\omega^2 \mapsto 3.$$

Then we can compute precisely the fixed fields of the subgroups $H$ of $G$: The subgroup $A_3$ has index 2 in $\mathcal{S}_3$ and its fixed field is $\mathbf{Q}(\omega)$, which has degree 2 over $\mathbf{Q}$;
the fixed field of $\langle(23)\rangle$ is $\mathbf{Q}(\sqrt[3]{2})$;
the fixed field of $\langle(13)\rangle$ is $\mathbf{Q}(\sqrt[3]{2}\omega)$;
the fixed field of $\langle(12)\rangle$ is $\mathbf{Q}(\sqrt[3]{2}\omega^2)$;

• **The separability issue**.
A field $\mathbf{K}$ is said to have $char(\mathbf{K}) = p$ (necessarily a prime number) if

$$\underbrace{1 + 1 + \ldots + 1}_{p \text{ times}} = 0.$$

Examples of such fields $\mathbf{K}$ are:
$\mathbf{Z}_p$, $\mathbf{Z}_p(x) = \{\frac{f}{g} \mid f, g \in \mathbf{Z}_p[x],\ g \neq 0\}$, $\mathbf{Z}_p[x]/(f)$, with $f \in \mathbf{Z}_p[x]$ irreducible.

*Let $\mathbf{K}$ be a field of characteristic $p$. Then there exist inseparable extensions of $\mathbf{K}(T)$.*
**Example.** Let $\mathbf{K}$ be the field $\mathbf{Z}_p(T)$. Consider the monic polynomial

$$f(x) = x^p - T \in \mathbf{Z}_p(T)[x].$$

- $f$ is irreducible: this follows by Eisenstein criterion with respect to the prime $T \in \mathbf{Z}_p[T]$, and then by Gauss lemma.
-$f$ is not separable: let $\alpha$ be a zero of $f$, i.e. an element $\alpha = \sqrt[p]{T}$, in some extension $\mathbf{L}$ of $\mathbf{K}$. But in characteristic $p$ one has $x^p - T = x^p - \alpha^p = (x - \alpha)^p$. In other words, all roots of $f$ in $\mathbf{L}$ are the same.
As a consequence, the field $\mathbf{L}$ is an inseparable extension of $\mathbf{K}$: it contains the element $\alpha$, whose minimal polynomial $x^p - T$ is not separable.