Due Tuesday July 16, 2019, at 24:00 Kathmandu time

1. *Consider the field* $K = \mathbf{Q}(\zeta_5, \sqrt[3]{2})$.
   (a) Show that K *admits subfields of degrees* 3 *and* 4.
   (b) Determine $[K : \mathbf{Q}]$.

Sol.: (a) The field $\mathbf{Q}(\zeta_5)$ is a subfield of degree 4 and the field $\mathbf{Q}(\sqrt[3]{2})$ is a subfield of degree 3. (b) the degree $[K : \mathbf{Q}]$ is divisible by the degrees of its subfields. Therefore $[K : \mathbf{Q}]$ is divisible by 12. On the other hand $[K : \mathbf{Q}] = [K : \mathbf{Q}(\zeta_5)] \cdot [\mathbf{Q}(\zeta_5) : \mathbf{Q}]$. Since the minimum polynomial of $\sqrt[3]{2}$ over $\mathbf{Q}(\zeta_5)$ divides $X^3 - 2$, we have that $[K : \mathbf{Q}(\zeta_5)] \leqslant 3$. Therefore $[K : \mathbf{Q}] \leqslant 3 \cdot [\mathbf{Q}(\zeta_5) : \mathbf{Q}] = 3 \cdot 4 = 12$. It follows that $[K : \mathbf{Q}] = 12$.

2.*Consider the field* $K = \mathbf{Q}(\zeta_{12}, \sqrt[4]{3})$.
   (a) Show that K *admits at least two distinct subfields of degree* 4.
   (b) Determine $[K : \mathbf{Q}]$.

Sol.: The field K admits the subfields $\mathbf{Q}(\zeta_{12})$ and $\mathbf{Q}(\sqrt[4]{3})$, each of degree 4 over $\mathbf{Q}$. (b) The minimum polynomial of ζ_{12} over $\mathbf{Q}$ is $X^4 - X^2 + 1$. Therefore the square of $w = \zeta_{12} + \zeta_{12}^{-1} \in \mathbf{Q}(\zeta_{12})$ is equal to $w^2 = \zeta_{12}^2 + 2 + \zeta_{12}^{-2} = 1 + 2 = 3$. It follows that w is a square root of 3 and that $\sqrt[4]{3}$ is a root of $X^2 - w$. Therefore the degree of $K = \mathbf{Q}(\zeta_{12}), \sqrt[4]{3})$ is $[\mathbf{Q}(\zeta_{12}) : \mathbf{Q}] \cdot [K : \mathbf{Q}(\zeta_{12})]$ is at most $4 \cdot 2 = 8$. We show that it is equal to 8 by showing that $\sqrt[4]{3} \notin \mathbf{Q}(\zeta_{12})$. Indeed, if $\sqrt[4]{3} \in \mathbf{Q}(\zeta_{12})$, then consideration of the degree shows that $\mathbf{Q}(\zeta_{12})$ is actually *equal* to $\mathbf{Q}(\sqrt[4]{3})$. But that is impossible. For instance, because $\mathbf{Q}(\sqrt[4]{3})$ can be embedded into $\mathbf{R}$, while $\mathbf{Q}(\zeta_{12})$ cannot.

3.*Consider the field* $K = \mathbf{Q}(\zeta_{13})$.
   How many distinct subfields of K *are there?*

Sol.: By Galois theory there are as many subfields of $\mathbf{Q}(\zeta_{13})$ as there are subgroups of $\mathbf{Z}_{13}^*$. Since $\mathbf{Z}_{13}^*$ is cyclic of order 12, there are as many subgroups as there are divisors of 12. In other words, there are $\#\{1, 2, 3, 4, 6, 12\} = 6$ subgroups.