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In this lecture we discussed further properties of separable extensions, defined finite Galois extensions and gave some examples of these.

First we proved the following theorem which extends Theorem 5 (Theorem 10.2 of the book) that we discussed in the last class to an arbitrary finite extension (need not be simple).

Theorem 1. *Suppose that $K' : K$ is a finite extension of degree d, and that $j : K \longrightarrow L$ is a monomorphism. If $K' : K$ is separable and $j(m_\alpha)$ splits over L for each α in K', then there are exactly d monomorphisms from K' to L extending j; otherwise, there are fewer than d such monomorphisms.*

As a consequence we proved the following results.

Corollary 2. *Suppose that $L : K$ is finite and that $L = K(\alpha_1, \dots, \alpha_r)$. If α_i is separable over $K(\alpha_1, \dots, \alpha_{i-1})$ for $1 \leq i \leq r$, then $L : K$ is separable.*

The following result states that to check that if a finite extension $K(\alpha_1, \dots, \alpha_r) : K$ is separable, it is enough to check each α_i is separable over K.

Corollary 3. *Suppose that $L : K$ is finite and that $L = K(\alpha_1, \dots, \alpha_r)$. If each α_i is separable over K then $L : K$ is separable.*

Corollary 4. *Suppose that $f \in K[X]$ is separable over K and that $L : K$ is a splitting field extension for f. Then $L : K$ is separable.*

Corollary 5. *Suppose that $L : K$ is finite, and M is an intermediate field. If $L : M$ and $M : K$ are separable, then so is $L : K$.*

We then defined finite Galois extensions.

Definition 6. A finite extension $L : K$ is said to be Galois if (i) $L : K$ is normal; (ii) $L : K$ is separable.

Example 7. (1) Since the extension $\mathbb{Q}(\sqrt{2}) : \mathbb{Q}$ is finite, normal and separable, it is Galois.
(2) Consider the finite extension $\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}$. Since the minimal polynomial of $\sqrt[3]{2}$ is $X^3 - 2$ which is separable, $\sqrt[3]{2}$ is separable. Therefore $\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}$ is separable. However, it is not normal because the irreducible polynomial $X^3 - 2$ does not split over $\mathbb{Q}$. Hence $\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}$ is not Galois.
(3) Consider $\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}$ where ω is a cube root of unity. Since $\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}$ is a splitting field extension of $X^3 - 2$ over $\mathbb{Q}$, it is normal. As $\sqrt[3]{2}, \omega$ are separable, the extension $\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}$ is also separable. Hence the extension $\mathbb{Q}(\sqrt[3]{2}, \omega) : \mathbb{Q}$ is Galois.

As a consequence of the results on separable and normal extensions that we discussed, we obtain the following result.

Theorem 8. *Suppose that $L : K$ is a finite Galois extension. Then there are $L : K$ automorphisms of L which fix K; otherwise there are fewer than $L : K$ such automorphisms.*