NAP 2019, MODULE-III, LECTURE 6: JUNE 11, 2019

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* Since we are leaving in the morning of June 14 (Friday), we took a class on June 11 (Tuesday) instead of June 14.

In this lecture we discussed further properties of normal extensions. First we completed a proof of the following theorem and then discussed its consequences.

Theorem 1. *An extension $L : K$ is normal if and only if it is a splitting field extension for some $S \subseteq K[X]$.*

If $L : K$ is finite, then S can be chosen to be a singleton set.

Corollary 2. *A finite extension $L : K$ is normal if and only if $L : K$ is a splitting field extension for some $g \in K[X]$.*

We defined a normal closure and proved its existence in the case $L : K$ is finite.

Definition 3. Suppose that $L : K$ is an algebraic. An extension $N : L$ is a *normal closure* for $L : K$ if $N : K$ is normal and if $L \subseteq L' \subseteq N$ such that $L' : K$ is normal, then $N = L'$.

Theorem 4. *If $L : K$ is finite, then it has a normal closure.*

Example 5. Consider an extension $L : K$ where $L = \mathbb{Q}(\sqrt[3]{2})$ and $K = \mathbb{Q}$. Then $\mathbb{Q}(\sqrt[3]{2}, \omega)$ where ω is a cube root of unity is a normal closure of $L : K$.

Suppose $L : K$ is a normal extension and M is an intermediate field. Is $L : M$ normal ? What can you say about $M : K$? The following corollary gives an affirmative answer to the first question.

Corollary 6. *If $L : K$ is normal and M is an intermediate field, then $L : M$ is normal.*

However, if $L : K$ is normal and M is an intermediate field, then $M : K$ need not be normal. We gave an example to illustrate this.

Example 7. Let $L = \mathbb{Q}(\sqrt[3]{2}, \omega)$ where ω is a cube root of unity and $K = \mathbb{Q}$. Then $L : K$ is normal. Consider $K \subseteq M := \mathbb{Q}(\sqrt[3]{2}) \subseteq L$. Then $M : K$ is not normal.