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In this lecture we discussed one of the important properties of field extensions, namely *normal extensions*.

Definition 1. An extension $L : K$ is said to be normal if
(1) $L : K$ is algebraic;
(2) for every $\alpha \in L$, the minimal polynomial of α over K splits over L. Equivalently, every irreducible polynomial $f \in K[X]$ either splits over L or has no roots in L.

We discussed some examples and non-examples of normal extensions.

Example 2. (a) Every quadratic extension $L : K$ is normal. (An extension is quadratic if $[L : K] = 2$).
(b) Let $L = \mathbb{Q}(\sqrt[3]{2})$ and $K = \mathbb{Q}$. Then $L : K$ is not normal because the polynomial $X^3 - 2$ is irreducible over $\mathbb{Q}$, but its complex roots do not belong to L.

Remark 3. (1) The word "irreducible" in (2) of Definition 1 is necessary. For example, let $L = \mathbb{Q}(\sqrt{2})$ and $K = \mathbb{Q}$. Then $L : K$ is normal. Consider $f(X) = (X^2 - 2)(X^2 - 3)$. Then $f(X)$ has a root in L, but it does not split in L.
(2) The extension $L : K$ in Definition 1 need not be finite.

Suppose $L : K$ is a splitting field extension of $f \in K[X]$ over K. Then $L = K(r_1, \ldots, r_n)$ where r_i are the roots of f. Is $L : K$ normal ? Clearly, $L : K$ is algebraic. Moreover, for each r_i, the minimal polynomial of r_i over K splits in L. But we need to verify (2) of Definition 1 for each $\alpha \in L$. In the next theorem we will prove that the extension $L : K$ is indeed a normal extension. For this we need to extend the definition of a splitting field.

Definition 4. Let $S \subseteq K[X]$ be a subset (S need not be finite). We say $L : K$ is a splitting field extension for S if
(1) every polynomial $f \in S$ splits over L;
(2) if $K \subseteq L' \subseteq L$ and each polynomial $f \in S$ splits in L', then $L = L'$.

We stated the following theorem and proved "only if" part of this.

Theorem 5. *An extension $L : K$ is normal if and only if it is a splitting field extension for some $S \subseteq K[X]$.*

Remark 6. In general, verifying an extension is normal or not is a difficult task. Theorem 5 can be effective to prove that an extension is normal in some cases. For example, let $L = \mathbb{Q}(\sqrt[3]{2}, \omega)$ where ω is a cube root of unity and $K = \mathbb{Q}$. Then $L : K$ is normal because L is a splitting field of $X^3 - 2$.

Some books, define normal extensions using Theorem 5. Namely, an extension $L : K$ is normal if L is a splitting field extension for some subset S in $K[X]$.