NAP 2019, MODULE-III, LECTURE # 2: JUNE 6, 2019

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Recall that In Lecture #1, we discussed the existence of a splitting field extension $L : K$ for any polynomial $f(X) \in K[X]$. We also know that the degree of a splitting field is at most $n!$ where n is the degree of the polynomial.

Today's Aim: Our aim today is show that a splitting field of a given polynomial is *essentially unique* in the sense that two splitting field extensions for a given polynomial are isomorphic, but such an isomorphism is not unique. This will be clear during the discussion.

We discussed following theorems.

Theorem 1. *Let $K(\alpha) : K$ is a simple algebraic extension and let the minimal polynomial of α be m_α. Let $i : K \to L$ be a monomorphism (or field homomorphism) from K into a field L, and $\beta \in L$.Then*

$$\boxed{\begin{array}{l}\exists \text{ a monomorphism } j : K(\alpha) \to L \\ \text{such that } j|_K = i \text{ and } j(\alpha) = \beta\end{array}} \Leftrightarrow \boxed{i(m_\alpha)(\beta) = 0}.$$

Moreover if there exists a monomorphism j then it is unique.

The proof of this theorem was discussed in great detail. The following two corollaries were also discussed.

Corollary 1. *Let $K(\alpha) : K$ and $K'(\alpha') : K'$ be simple algebraic extensions. Let $i : K \to K'$ be an isomorphism. Then there exists an isomorphism $j : K(\alpha) \to K'(\alpha')$ with $j(\alpha) = \alpha'$ and $j|_K = i$ if and only if $i(m_\alpha) = m_{\alpha'}$. If j exists, it is unique.*

Corollary 2. *Let $K(\alpha) : K$ is a simple algebraic extension. Let $i : K \to L$ be a monomorphism and $i(m_\alpha)$ has r distinct roots in L. Then there are exactly r distinct monomorphisms $j : K(\alpha) \to L$ with $j|_K = i$.*

Theorem 2. *Let $\Sigma : K$ be a splitting field extension for a polynomial $f(X) \in K[X]$, and $i : K \to L$ be a monomorphism from K into a field L. Then*

$$\boxed{\exists \text{ a monomorphism } j : \Sigma \to L \text{ with } j|_K = i} \Leftrightarrow \boxed{i(f(X)) \text{ splits over } L.}$$

The proof of this was done by using induction on the degree of the polynomial $f(X)$.

The following two corollaries were mentioned but not discussed in detail which will continue in the next lecture.

Corollary 3. *Let $iK \to K'$ be an isomorphism of fields and $f(X) \in K[X]$. Let $\Sigma : K$ be a splitting field extension for $f(X)$ and $\Sigma' : K'$ a splitting field for $i(f(X))$. Then there exists an isomorphism $j : \Sigma \to \Sigma'$ such that $j|_K = i$.*

Corollary 4. *Let $f(X) \in K[X]$ be an irreducible polynomial and $\Sigma : K$ a splitting field extension for $f(X)$. Let α, β be roots of f in Σ. Then there exists an automorphism $\sigma : \Sigma \to \Sigma$ such that $\sigma(\alpha) = \beta$. and $\sigma|_K$ is the identity map on K.*