NAP 2019, MODULE-III, LECTURE # 1: JUNE 3, 2019

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Theorem 1 (Fundamental theorem of Algebra - FTA). *Let $f(X) \in \mathbb{C}[X]$ be a non-constant polynomial over the field $\mathbb{C}$. Then $f(X)$ has a root in $\mathbb{C}$. Equivalently, all the roots of $f(X)$ are in $\mathbb{C}$.*

Note that $\mathbb{Q}[X] \subset \mathbb{C}[X]$, since $\mathbb{Q} \subset \mathbb{C}$. For $f(X) \in \mathbb{Q}[X]$, we have $f(X) \in \mathbb{C}[X]$ then by FTA there exists $\alpha_1, \cdots, \alpha_n \in \mathbb{C}$ and $\lambda \in \mathbb{Q}$ such that

$$f(X) = \lambda(X - \alpha_1) \cdots (X - \alpha_n).$$

Note that each α_i is algebraic over $\mathbb{Q}$, since $f(\alpha_i) = 0$. Take $L = \mathbb{Q}(\alpha_1, \cdots, \alpha_n)$ then $[L : \mathbb{Q}] < \infty$. The polynomial $f(X)$ factorizes into linear factors over the field L.

Remark: For a given polynomial $f(X) \in \mathbb{Q}[X]$ if we wish to factorize it into linear factors then there is a finite algebraic extension $L : \mathbb{Q}$ over which it can be done. All of $\mathbb{C}$ is not needed for this purpose. Of course, the field L depends on the given polynomial $f(X)$. We coud show the existence of L for a given polynomial $f(X) \in \mathbb{Q}[X]$ because of the FTA.

Today's Aim: Our aim is to construct a finite algebraic extension $L : K$ for a given $f(X) \in K[X]$ such that $f(X)$ factorizes into linear factors over L.

Definition 2. Let K be a field and $L : K$ an extension. Let $f(X) \in K[X]$. We say $f(X)$ **splits over** L if we can write

$$f(X) = \lambda(X - \alpha_1) \cdots, (X - \alpha_n)$$

where $\alpha_1, \cdots, \alpha_n \in L$ and $\lambda \in K$.

Definition 3. Let K be a field and $L : K$ an extension. Let $f(X) \in K[X]$. We say that $L : K$ **is a splitting field extension for** $f(X)$ **over** K (or L is a splitting field for $f(X)$ when K is clear from the context) if

(a) $f(X)$ splits over L and,
(b) there is no proper subfield $L' \subset L$ containing K such that $f(X)$ splits over L'.

Example 4. Consider $X^2 - 2 \in \mathbb{Q}[X]$. Then $\mathbb{Q}(\sqrt{2}) : Q$ is a splitting field extension for $X^2 - 2$. Note that the $X^2 - 2$ splits over $\mathbb{Q}(\sqrt{2}, \sqrt{3}), \mathbb{Q}(\sqrt{2}, \pi), \mathbb{R}, \mathbb{C}$ etc. but these are not the splitting field extension for $X^2 - 2 \in \mathbb{Q}[X]$.

We achieve our aim by discussing the theorems below with proof.

Theorem 5. *Suppose that $L : K$ is an extension and that a polynomial $f(X) \in K[x]$ splits over L as $f(X) = \lambda(X - \alpha_1) \cdots, (X - \alpha_n)$. Then $K(\alpha_1, \cdots, \alpha_n)$ is a splitting field for $f(X)$.*

Theorem 6. *Suppose that $f(X) \in K[x]$ is irreducible polynomial of degree n. Then there is a simple algebraic extension $K(\alpha) : K$ such that $[K(\alpha) : K] = n$ and $f(\alpha) = 0$.*

Theorem 7. *Suppose that $f(X) \in K[x]$. Then there exists a splitting field extension $L : K$ for $f(X)$, with $[L : K] \leq n!$.*