NAP 2019, MODULE-III, SOLUTIONS OF HOMEWORK ASSIGNMENT # 1

SHREEDEVI MASUTI & SHIV PRAKASH PATEL

Comment: In case, you notice some typo or inaccuracy please let us know. **Problem 7.1 on page 59:**
Let $f(X) \in \mathbb{R}[X]$ be a non-constant irreducible polynomial. By Fundamental theorem of algebra $f(X)$ has all its roots in $\mathbb{C}$.

Case 1: One of the roots of $f(X)$ is in $\mathbb{R}$, say α. Then $X - \alpha \in \mathbb{R}[X]$ is a factor of $f(X)$. But, $f(X)$ is irreducible and therefore $f(X) = \lambda(X - \alpha)$ which is of degree 1.

Case 2: None of the roots of $f(X)$ are in $\mathbb{R}$. Let $a + ib \in \mathbb{C}$ be a root of $f(X)$ where $a, b \in \mathbb{R}$ and $b \neq 0$ (as usual $i = \sqrt{-1}$). Then (a hint was given in the class) that $a - ib \in \mathbb{C}$ is also a root of $f(X)$ [some details are needed here]. Since $b \neq 0$, $a + ib \neq a - ib$ therefore $(X - (a + ib))(X - (a - ib))$ is a factor of $f(X)$. Therefore $X^2 - 2aX + a^2 + b^2 \in \mathbb{R}[X]$ is a factor of $f(X)$. But, $f(X)$ is irreducible in $\mathbb{R}[X]$ therefore $f(X) = \lambda(X^2 - 2aX + a^2 + b^2)$ for some $\lambda \in \mathbb{R}$ which is of degree 2.

Problem 7.2 on page 62:
Want to show that $f(X) = X^3 - X + 1 \in \mathbb{Z}_3[X]$ is irreducible. Note that if $f(X)$ is reducible then at least one of the factors will be of degree one, that is, $f(X)$ will have a root in $\mathbb{Z}_3$. Since $\mathbb{Z}_3$ has only 3 elements we check that $f(0) = 1 \neq 0, f(1) = 1 \neq 0, f(2) = 1 \neq 0$. We find that $f(X)$ has no root in $\mathbb{Z}_3$ and therefore it is irreducible.

By Theorem 7.2 there is a simple extension $\mathbb{Z}_3(\zeta) : \mathbb{Z}_3$ of degree 3 such that $f(\zeta) = 0$. Then, of course, $\zeta + 1, \zeta - 1 \in \mathbb{Z}_3(\zeta)$. We find that

$$f(\zeta+1) = (\zeta+1)^3 - (\zeta+1) + 1 = \zeta^3 + 3\zeta(\zeta+1) + 1 - \zeta - 1 + 1 = \zeta^3 - \zeta + 1 = 0,$$
$$f(\zeta-1) = (\zeta-1)^3 - (\zeta-1) + 1 = \zeta^3 - 3\zeta(\zeta-1) - 1 - \zeta + 1 + 1 = \zeta^3 - \zeta + 1 = 0$$

(since $3 = 0$ in $\mathbb{Z}_3$ and $f(\zeta) = 0$). Note that all three elements $\zeta, \zeta+1, \zeta-1$ are distinct. Therefore $f(X) = X^3 - X + 1 = (X - \zeta)(X - \zeta - 1)(X - \zeta + 1) \in \mathbb{Z}_3(\zeta)[X]$, that is $f(X)$ splits over $\mathbb{Z}_3(\zeta)$.

Note that for any field K if $\mathbb{Z}_3 \subset K \subset \mathbb{Z}_3(\zeta)$ then either $K = \mathbb{Z}_3$ or $K = \mathbb{Z}_3(\zeta)$. But $f(X)$ has no root in $\mathbb{Z}_3$, thus there is no proper subfield K of $\mathbb{Z}_3(\zeta)$ such that $f(X)$ splits over K. Therefore $\mathbb{Z}_3(\zeta)$ is a splitting field for $f(X)$.

Since $[\mathbb{Z}_3(\zeta) : \mathbb{Z}_3] = 3$, $\mathbb{Z}_3(\zeta) = 3^3 = 27$. We skip writing the multiplication table.

Problem 7.3 on page 62:
Note that the simple transcendental extension $K(t)$ of K is the quotient field of the the polynomial ring $K[t]$ which is a unique factorization domain (UFD). Note that $X^n - t \in K[t][X]$ as well as in $K(t)[X]$. By Gauss lemma,

$$X^n - t \text{ is irreducible over } F[t] \text{ if and only if it is irreducible over } F(t).$$

As an application to the Exercise 5.4, one can prove that $X^n - t$ is irreducible in $K[t][X] = K[X, t]$. We prove this below.

Method 1 for irreducibility: Let $X^n - t = F(X,t)G(X,t)$ where $F(X,t), G(X,t) \in K[X,t]$. Since t-degree of the polynomial $X^n - t$ is 1 exactly one of the polynomial $F(X,t), G(X,t)$ has t-degree

1 and the other has t-degree 0. Say $F(X,t) = f(x)$ and $G(X,t) = g(X)t + h(X)$. Then we have

$$X^n - t = f(X)(g(X)t + h(X)) = f(X)g(X)t + f(X)h(X).$$

Thus we get $f(X)g(X) = -1$, i.e. $f(X)$ is invertible in $K[X]$, i.e. $f(X) \in K^*$. Thus $F(X) = f(X)$ is a unit. This prove that $X^n - t$ is irreducible in $K[X,t]$.

Method 2 for irreducibility: Since t is prime (or irreducible) in $K[t]$ we use Eisenstein criterion to conclude that $X^n - t$ is irreducible. Note that t divides all the coefficients of $X^n - t$ except the leading coefficient but t^2 does not divide the constant term $-t$.

Since $f(X) = X^n - t \in K(t)[X]$ is irreducible. Let s be a root of $f(X)$ is an extension $K(t)(s) : K(t)$ which is of degree n (by Theorem 7.2). First observation is that $s^n = t$ therefore $K(t)(s) = K(s)$. If s' is another root of $f(X)$ then $(s'/s)^n = t/t = 1$, that is s'/s is a root of $X^n - 1$ which has all its roots in K. Then $s' = \omega s$ where $\omega \in K$ is a root of $X^n - 1$. Thus the set of all the roots of $X^n - t$ is $S = \{s\omega : \omega \text{ is a root of } X^n - 1\}$. Since roots of $X^n - 1$ are in K, as splitting field of $f(X)$ is

$$K(t)(S) = K(t)(s) = K(s).$$

The extension $K(s) : K(t)$ has $s^n = t$ and $K(t) \to K(s)$ is given by fixing K and $t \mapsto s^n$.

Problem 7.4 on page 67: Given that $f(X) = X^4 - 2X^3 + 7X^2 - 6X + 12 \in \mathbb{Q}[X]$ and $i\sqrt{3}, 1 + i\sqrt{3}$ are roots of $f(X)$. Let $L : \mathbb{Q}$ be a splitting field extension for $f(X)$. Assume that $\sigma : L \to L$ is an automorphism such that $\sigma(i\sqrt{3}) = 1 + i\sqrt{3}$. Then

$$\sigma(-3) = \sigma(i\sqrt{3} \cdot i\sqrt{3}) = \sigma(i\sqrt{3})\sigma(i\sqrt{3}) = (1 + i\sqrt{3})(1 + i\sqrt{3}) = -2 + 2i\sqrt{3}.$$

But σ is a field automorphism $\sigma(1) = 1 \Rightarrow \sigma(-3) = -3 \neq -2 + 2i\sqrt{3}$, a contradiction. Thus no σ is possible with desired property.

Problem 7.5 on page 70:

Suppose that $M : L$ and $L : K$ are extensions, and that $\alpha \in M$ is algebraic over K. It is NOT always the case that $[L(\alpha) : L]$ divides $K(\alpha) : K]$.

[Heuristic idea: *Let $m_K(X) \in K[X]$ and $m_L(X) \in L[X]$ be the minimal polynomial of α over K and L respectively. Then there is a $g(X) \in L[X]$ such that $m_K(X) = m_L(X) \cdot g(X)$. Note that*

$$[K(\alpha) : K] = \text{degree}(m_K(X)) \text{ and } [L(\alpha) : L] = \text{degree}(m_L(X)).$$

Moreover, $\text{degree}(m_K(X)) = \text{degree}(m_L(X)) + \text{degree}(g(X))$. *There is no reason why* $\text{degree}(m_L(X))$ *should divide* $\text{degree}(m_K(X))$. Here is an example.]

Consider $f(X) = X^5 - 20X + 2 \in \mathbb{Q}[X]$. By Eisenstein criterion $f(X)$ is irreducible. Use real analysis (maxima, minima principle etc.) to say that $f(X)$ has exactly 3 roots in $\mathbb{R}$ and 2 roots in $\mathbb{C}$ which are not in $\mathbb{R}$. Write $\alpha_1, \alpha_2, \alpha_3$ be roots in $\mathbb{R}$ and $\alpha, \bar{\alpha}$ the roots in $\mathbb{C}$ which are not in $\mathbb{R}$. Here $\bar{\alpha}$ is the complex conjugate of α and we have seen in Exercise 7.1 that if α is a root of a polynomial over real numbers then $\bar{\alpha}$ is also a root. Then

$$f(X) = (X - \alpha_1)(X - \alpha_2)(X - \alpha_3)(X - \alpha)(X - \bar{\alpha}).$$

Note that $f(X)$ is a minimal polynomial for all the elements $\alpha_1, \alpha_2, \alpha_3, \alpha, \bar{\alpha}$ over $\mathbb{Q}$.

Now take $K = \mathbb{Q}, L = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ and $M = \mathbb{C}$. Take $\alpha \in \mathbb{C}$.

Then $m_K(X) = m_{\mathbb{Q}}(X) = f(X)$ therefore $[K(\alpha) : K] = \text{degree}(f(X)) = 5$.

Note that $f(X)$ has a root in L, it is not irreducible over L. Then $f(X)$ is not the minimal polynomial for α over L. Note that $g(X) = (X-\alpha)(X-\bar{\alpha}) = \dfrac{f(X)}{(X-\alpha_1)(X-\alpha_2)(X-\alpha_3)} \in L[X]$ is irreducible because it has no roots in L (since $L \subset \mathbb{R}$ but $\alpha, \bar{\alpha} \notin \mathbb{R}$). Therefore the minimal polynomial of α over L is $m_L(X) = g(X)$ which is of degree 2. Therefore $[L(\alpha) : L] = \text{degree}(g(X)) = 2$. But 2 does not divide 5.

Problem 7.6 on page 70:

For a 3 degree polynomial over $\mathbb{Z}_2 = \{0,1\}$, it is reducible if and only if it has a root in $\mathbb{Z}_2$. Here is the list of all monic cubic polynomials in $\mathbb{Z}_2[X]$ and their factorization.

(a) $X^3 = X \cdot X \cdot X$.
(b) $X^3 + 1 = (X+1)(X^2+X+1)$.
(c) $X^3 + X = X(X^2+1) = X(X+1)(X+1)$.
(d) $f(X) := X^3 + X + 1$ is irreducible (since no roots in $\mathbb{Z}_2$).
(e) $X^3 + X^2 = X^2(X+1)$.
(f) $g(X) := X^3 + X^2 + 1$ is irreducible (since no roots in $\mathbb{Z}_2$)
(g) $X^3 + X^2 + X = X(X^2+X+1)$.
(h) $X^3 + X^2 + X + 1 = (X^2+1)(X+1) = (X+1)^3$.

For the polynomials in $(a), (c), (e)$ and (h) the field $\mathbb{Z}_2$ is a splitting field since all the roots of these polynomials are in $\mathbb{Z}_2$.

For the polynomials in (b) and (g) one root is there in $\mathbb{Z}_2$ but the roots of X^2+X+1 are not in $\mathbb{Z}_2$. By Example 4 discussed in the book, its splitting field is a two degree extension which we can write as $\mathbb{Z}_2(\alpha) = \{0, 1, \alpha, 1+\alpha\}$. Thus the splitting fields for X^3+1 and X^3+X^2+X are isomorphic.

Using Theorem 7.2, for the polynomial $f(X) = X^3+X+1 \in \mathbb{Z}_2[X]$ which is irreducible, let α be a root in an extension $\mathbb{Z}_2(\alpha) : \mathbb{Z}_2$ which is of degree 3. Then

$$\mathbb{Z}_2(\alpha) = \{0, 1, \alpha, 1+\alpha, \alpha^2, 1+\alpha^2, \alpha+\alpha^2, 1+\alpha+\alpha^2\}.$$

Then we find that $f(X) = X^3+X+1 = (X-\alpha)(X-\alpha^2)(X-\alpha-\alpha^2)$, i.e. all the roots of $f(X)$ are in $\mathbb{Z}_2(\alpha)$ and there is no proper subfield where $f(X)$ splits into linear polynomials. Thus $\mathbb{Z}_2(\alpha) : \mathbb{Z}_2$ is a splitting field extension for $f(X)$.

Now consider $g(X) = X^3+X^2+1 \in \mathbb{Z}_2[X]$ which is irreducible. Let α be a root of $f(X)$ considered above. Then verify that

$$g(1+\alpha) = (1+\alpha)^3 + (1+\alpha)^2 + 1 = 0.$$

Since $\alpha, \alpha^2, \alpha+\alpha^2$ are roots of $f(X)$, the roots of $g(X)$ are $1+\alpha, 1+\alpha^2, 1+\alpha+\alpha^2$ in $\mathbb{Z}_2(\alpha)$. Therefore

$$g(X) = X^3+X^2+1 = (X-1-\alpha)(X-1-\alpha^2)(X-1-\alpha-\alpha^2).$$

Thus $\mathbb{Z}_2(\alpha) : \mathbb{Z}_2$ is also a splitting field extension for $g(X)$. Therefore splitting fields for $f(X)$ and $g(X)$ are isomorphic.

Problem 7.7 on page 70:

Let $f(X) = X^4 - 5X^2 + 6, g(X) = X^4 + 5X^2 + 6$ and $h(X) = X^4 - 5$ in $\mathbb{Q}[X]$. We will find a splitting filed extension for all these polynomials. By fundamental theorem of algebra, these polynomials split over $\mathbb{C}$. The linear factors over $\mathbb{C}$ are as follows:

(a) $f(X) = (X^2-2)(X^2-3) = (X+\sqrt{2})(X-\sqrt{2})(X+\sqrt{3})(X-\sqrt{3})$,
(b) $g(X) = (X^2+2)(X^2+3) = (X+i\sqrt{2})(X-i\sqrt{2})(X+i\sqrt{3})(X-i\sqrt{3})$,
(c) $h(X) = (X^2-\sqrt{5})(X^2+\sqrt{5}) = (X-5^{\frac{1}{4}})(X+5^{\frac{1}{4})}(X-i\,5^{\frac{1}{4}})(X+i\,5^{\frac{1}{4}})$.

By Theorem 7.1, a splitting field for these polynomials $f(X), g(X)$ and $h(X)$ is given by

(a) $\mathbb{Q}(\sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$,
(b) $\mathbb{Q}(i\sqrt{2}, -i\sqrt{2}, i\sqrt{3}, -i\sqrt{3}) = \mathbb{Q}(i\sqrt{2}, i\sqrt{3})$ and
(c) $\mathbb{Q}(5^{\frac{1}{4}}, -5^{\frac{1}{4}}, i\,5^{\frac{1}{4}}, -i\,5^{\frac{1}{4}}) = \mathbb{Q}(i, 5^{\frac{1}{4}})$

respectively.

The degrees are as follows (justify each equality below):

(a) $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] \cdot [\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2 \cdot 2 = 4$,
(b) $[\mathbb{Q}(i\sqrt{2}, i\sqrt{3}) : \mathbb{Q}] = [\mathbb{Q}(i\sqrt{2}, i\sqrt{3}) : \mathbb{Q}(i\sqrt{2})] \cdot [\mathbb{Q}(i\sqrt{2}) : \mathbb{Q}] = 2 \cdot 2 = 4$ and
(c) $[\mathbb{Q}(i, 5^{\frac{1}{4}}) : \mathbb{Q}] = [\mathbb{Q}(i, 5^{\frac{1}{4}}) : \mathbb{Q}(5^{\frac{1}{4}})] \cdot [\mathbb{Q}(5^{\frac{1}{4}}) : \mathbb{Q}] = 2 \cdot 4 = 8$.

Moreover, we have the following (justify each of these):

(a) $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$,
(b) $\mathbb{Q}(i\sqrt{2}, i\sqrt{3}) = \mathbb{Q}(i\sqrt{2} + i\sqrt{3})$ and
(c) $\mathbb{Q}(i, 5^{\frac{1}{4}}) = \mathbb{Q}(i + 5^{\frac{1}{4}})$.

Problem 7.8 on page 70:

Write $f_1(X) = X^4 + 1, f_2(X) = X^4 + 4, f_3(X) = (X^4+1)(X^4+4)$ and $f_4(X) = (X^4-1)(X^4+4)$ in $\mathbb{Q}[X]$. As in Exercise 7.7, using Fundamental theorem of algebra. We first factor them over $\mathbb{C}$ and using Theorem 7.1 we find splitting fields. We find that

(a) $f_1(X) = X^4 + 1 = (X^2+i)(X^2-i) = (X - \frac{1+i}{\sqrt{2}})(X - \frac{1-i}{\sqrt{2}})(X - \frac{-1+i}{\sqrt{2}})(X - \frac{-1-i}{\sqrt{2}})$,
(b) $f_2(X) = X^4 + 4 = (X^2+2i)(X^2-2i) = (X-1-i)(X-1+i)(X+1+i)(X+1-i)$,
(c) $f_3(X) = f_1(X)f_2(X)$ which is product of linear factors in $f_(X)$ and $f_2(X)$,
(d) $f_4(X) = (X^4-1)f_2(X) = (X-1)(X+1)(X-i)(X+i)f_2(X)$ and we already have linear factors of $f_2(X)$.

We write L_1, L_2, L_3 and L_4 a splitting field for $f_1(X), f_2(X), f_3(X)$ and $f_4(X)$ respectively. We have

(a) $L_1 = \mathbb{Q}(\frac{1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}) = \mathbb{Q}(\frac{1}{\sqrt{2}}, i) = \mathbb{Q}(\sqrt{2}, i)$
(b) $L_2 = \mathbb{Q}(1+i, 1-i, -1+i, -1-i) = \mathbb{Q}(i)$
(c) $L_3 = \mathbb{Q}(\frac{1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, 1+i, 1-i, -1+i, -1-i) = \mathbb{Q}(\sqrt{2}, i)$ and
(d) $L_4 = \mathbb{Q}(1, -1, i, -i, 1+i, 1-i, -1+i, -1-i) = \mathbb{Q}(i)$.

There degrees are as follows:

(a) $[L_1 : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}(i)] \cdot [\mathbb{Q}(i) : \mathbb{Q}] = 2 \cdot 2 = 4$,
(b) $[L_2 : \mathbb{Q}] = [\mathbb{Q}(i) : \mathbb{Q}] = 2$,
(c) $[L_3 : \mathbb{Q}] = [\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}] = 4$ and
(d) $[L_4 : \mathbb{Q}] = [\mathbb{Q}(i) : \mathbb{Q}] = 2$.

Verify the following: $\mathbb{Q}(\sqrt{2}, i) = \mathbb{Q}(\sqrt{2} + i)$.

Problem 7.9 on page 70:
Let K be a field and let $f(X) \in K[X]$ be a monic polynomial of degree $n > 0$. Let $L : K$ be a splitting field extension for $f(X)$. We want to prove that $[L : K]$ divides $n!$. We prove this by induction on the degree$(f(X)) = n$.
If $n = 1$, $f(X)$ splits in K then $L = K$ and there is nothing to prove.
Assume the theorem true for all $n' > 0$ and $n' < n$.
Case 1: $f(X)$ is irreducible. By Theorem 7.2, $K(\alpha) \simeq K[X]/(f)$ is a field in which $f(X)$ has a root, and $[K(\alpha) : K] = n$. So $f(X) = (X - \alpha)g(X) \in K(\alpha)[X]$ for some $g(X) \in K(\alpha)[X]$ and degree$(g(X)) = n - 1$. By induction hypothesis, if $L : K(\alpha)$ is a splitting field for $g(X)$then $[L : K(\alpha)]$ divides $(n-1)!$. Then $[L : K] = [L : K(\alpha)] \cdot [K(\alpha) : K] = [L : K(\alpha)]n$ which divides $n!$.
Case 2: $f(X)$ is reducible.
Say that $f(X) = g(X)h(X)$, where degree$(g(X)) = r$ and degree$(h(X)) = s$, with $0 < r, s < n$ and $r + s = n$. By induction hypothesis, if $L_1 : K$ is a splitting field extension for $g(X)$ then $[L_1 : K]$ divides $r!$. Consider $h(X) \in L_1[X]$. If $L : L_1$ is a splitting field extension for $h(X)$ then $[L : L_1]$ divides $s!$ (by the induction hypothesis). Note that $f(X)$ splits in L. In fact, it is clear that $L : K$ is a splitting field extension for $f(X)$ since it is generated by all the roots of $f(X)$ which are precisely the roots of $h(X)$ and $g(X)$. Now the degree $[L : K] = [L : L_1] \cdot [L_1 : K]$, which divides $r!s!$. Since $n = r + s$, we know that $r!s!$ divides $n!$, so $[L : K]$ divides $n!$.

Problem 7.10 on page 70: Let $f(X) = X^3 - 5 \in K[X]$ where $K = \mathbb{Z}_7, \mathbb{Z}_{11}$ or $\mathbb{Z}_{13}$. Note that $f(X)$ is irreducible over K if and only if $f(X)$ has a root in$K$.

Case 1: $K = \mathbb{Z}_7$
We see that $f(0) = -5 \neq 0, f(1) = -4 \neq 0, f(2) = 3 \neq 0, f(3) = 1 \neq 0, f(4) = -32 = 3 \neq 0, f(5) = -13 = 1 \neq 0, f(6) = -6 = 1 \neq 0$ therefore $f(X)$ is irreducible over $\mathbb{Z}_7$. Let α be a root of $f(X)$ in an extension $\mathbb{Z}_7(\alpha) : \mathbb{Z}_7$ which has degree 3. Then we get that

$$f(X) = X^3 - 5 = (X - \alpha)(X - 2\alpha)(X - 4\alpha)$$

that is $f(X)$ splits over $\mathbb{Z}_7(\alpha)$. Therefore $\mathbb{Z}_7(\alpha) : \mathbb{Z}_7$ is a splitting field extension for $f(X)$.

Case 2: $K = \mathbb{Z}_{11}$
Note that $f(5) = 22 = 0$in $\mathbb{Z}_{11}$, therefore $f(X)$ is reducible whose one factor is $X - 3$. We get $f(X) = (X - 3)(X^2 + 3X - 2)$. We check that $g(X) := X^2 + 3X - 2$ is irreducible since it does not have a root in $\mathbb{Z}_{11}$. Checking: $g(0) = -2 \neq 0, g(1) = 2 \neq 0, f(2) = 8 \neq 0, f(3) = 5 \neq 0, f(4) = 4 \neq 0, f(5) = 5 \neq 0, f(6) = 8 \neq 0, f(7) = 2 \neq 0, f(8) = -2 \neq 0, f(9) = -4 \neq 0, f(10) = -4 \neq 0$. Note that a splitting field for $g(X)$ is also a splitting field for $f(X)$. Let β be a root of $g(X)$ in an extension $\mathbb{Z}_{11}(\beta) : \mathbb{Z}_{11}$ then $\mathbb{Z}_{11}(\beta)$ is a splitting field for $g(X)$. By Example 4 in the book $\mathbb{Z}_{11}(\beta)$ is isomorphic to $\mathbb{Z}_{11}(\sqrt{6})$ since 6 is the discriminant of $g(X)$.

Case 3: $K = \mathbb{Z}_{13}$
Check that $7, 8, 11 \in \mathbb{Z}_{13}$ are roots of $f(X) = X^3 - 5 \in \mathbb{Z}_{13}[X]$. Then

$$f(X) = (X - 7)(X - 8)(X - 11).$$

That is $f(X)$ splits over $\mathbb{Z}_{13}$ itself. Therefore $\mathbb{Z}_{13}$ itself is a splitting field for $f(X)$.