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• Finite extensions. Degree.

• Examples: take two fields K and L among the following ones

$$\mathbb{Q},\ \mathbb{R},\ \mathbb{C},\ \mathbb{Q}(\sqrt{2}),\ \mathbb{Q}(i),\ \mathbb{Q}(\sqrt[3]{2}),\ \mathbb{Q}(i,\sqrt{2})$$

such that L is an extension of K. Is $L : K$ a finite extension? If the answer is yes, give the degree and a basis of L over K.

• Which are the subfields of $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt[3]{2})$, $\mathbb{Q}(i,\sqrt{2})$?

• Tower law for the degree of field extensions. Examples. Proof of the law. Some consequences.

• Prime field: $\mathbb{Q}$, $\mathbb{Z}/p\mathbb{Z}$. These are the fields without proper subfield.

• Algebraic elements, transcendental elements.

• Example of a transcendental element: if $x \in K(T) \setminus K$, then x is transcendental over K (exercise 4.8 p. 47).

• Algebraic extension. Examples.

• Proof that a finite extension is algebraic.

• The field of algebraic numbers. This field is not a finite extension of $\mathbb{Q}$.

Fact : if α and β are algebraic numbers, with $\alpha = \alpha_1$ root of a polynomial in $\mathbb{Q}[X]$ with complex roots $\alpha_1, \dots, \alpha_d$, and with $\beta = \beta_1$ root of a polynomial in $\mathbb{Q}[X]$ with complex roots $\beta_1, \dots, \beta_m$, then the polynomials

$$\prod_{i=1}^{d}\prod_{j=1}^{m}(X - \alpha_i - \beta_j) \text{ and } \prod_{i=1}^{d}\prod_{j=1}^{m}(X - \alpha_i\beta_j)$$

have coefficients in $\mathbb{Q}$ (no proof yet) and roots $\alpha + \beta$ and $\alpha\beta$ respectively.

• The field $\mathbb{Q}(e^{2i\pi/p})$ with p prime.