NAP 2019, CLASS #4, MAY 10, 2019

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• We started Chapter 3 (Rings). We warned students that not everybody requires a ring to have at least two elements (but we'll go along with that convention now). We gave some examples and pointed out that $\mathbb{Z} \times 0$ is a ring with 1 but is not a subring of $\mathbb{Z} \times \mathbb{Z}$ (since $1_{\mathbb{Z}\times\mathbb{Z}} \notin \mathbb{Z} \times 0$). We ignored the overly pedantic definition of the polynomial ring, since everyone knows what a polynomial is. We defined the degree of a non-zero polynomial. Some students were bothered by the fact that 0 does not have a degree, so we said they could say the 0 polynomial has degree $-\infty$ if they like. We pointed out that in $\mathbb{Z}[x]$ the degree of a product is the sum of the degrees, but that this can fail, e.g., for $(\mathbb{Z}/(12))[x]$.

• We talked a little about homomorphisms and warned students that the author's term "epimorphism" is usually not used; more often "onto ring homomorphism" or "surjective ring homomorphism" is used (since epimorphisms in the category of rings are not necessarily surjective — a comment we really did not expect them to understand). We pointed out that $\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ via $n \mapsto (n, 0)$ is *not* a homomorphism but that the projection $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ onto the first coordinate *is* a homomorphism. We forgot to mention that the diagonal embedding $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, via $n \mapsto (n, n)$ is a homomorphism.

• We defined ideals and referred to the condition "$rx \in I\ \forall r \in R,\ x \in I$" as the "sponge" property. We said that ideals are the ring-theoretic analogue of normal subgroups and made the quotient group R/I into a ring in the obvious way. We defined the kernel, $\operatorname{Ker}\varphi$, of a ring homomorphism φ and did the First Isomorphism Theorem for Rings. (Not much in the way of a proof was necessary, of course, since the map on cosets had already been defined carefully in the context of groups.) We defined principal ideals.

• From now on we restrict to integral domains. If R is a domain, so is $R[x]$, and we have $\deg(fg) = \deg(f) + \deg(g)$ for non-zero polynomials f and g. We defined "irreducible element" and "unit" and pointed out that a non-zero element p is irreducible if and only if (p) is maximal among proper principal ideals. Also, u is a unit if and only if $(u) = R$. We defined the ACCPI —Ascending Chain Condition for Principal Ideals (and should have pointed out that the rest of the world calls this ACCP). We showed that ACCPI is equivalent to the condition that every non-empty collection of maximal ideals has a maximal element, and then used this to give the classical proof (and attributed the idea to Emmy Noether) that ACCPI implies that every non-zero non-unit is a product of irreducible elements. (Let $\mathcal{S}$ be the set principal ideals (b) for which b is a non-zero non-unit but is *not* a product of principal ideals. If $\mathcal{S} = \emptyset$ we are done. If not, choose a maximal element (b), note that b is not irreducible, and obtain a contradiction without even breaking a sweat.)

Our goal is to generalize the Fundamental Theorem of Arithmetic (FTA): Over the ring of integers $\mathbb{Z}$, every integer $n > 1$ can be expressed as a product of prime integers in a unique way (up to ordering of the prime factors). In the generalized version we will have a new definition of "prime", so we instead use the term "irreducible". We defined *associate* elements in an arbitrary integral domain and *unique factorization domain (UFD)*.

• Things we should have mentioned but did not: The condition that every non-zero non-unit is a product of irreducibles is called "atomic". We prvoed that ACCPI $\implies$ atomic, and an obvious question is whether the converse holds: Do atomic domains satisfy ACCPI? No. The first example was obtained by Anne Grams (a 1972 Ph.D. student of Robert Gilmer) in the early 1970s. [A. Grams, Atomic domains and the ascending chain condition for principal ideals, Proc. Cambridge Phil. Soc. 75 (1974), 321–329].

• **Reading Assignment for Week 2**: Garling, pp. 18–36, 47–53.

• **Homework Assignment** (due Monday, May 20, 10 pm Kathmandu time). Here are 14 problems; do 11 of them (so you may choose three of them to omit).

(1) pp. 23–24: 3.4, 3.7, 3.9
(2) p. 27: 3.10, 3.11, 3.12, 3.14
(3) p.29: 3.16, 3.17, 3.18
(4) pp. 30–31: 3.19
(5) pp. 34–35: 3.24, 3.25, 3.26

• **General Rule Concerning Homework**: You are expected to *prove* everything. This applies, for example, to Problem 1.13 on page 9 (on the homework due Monday, May 13). You are asked to decide which of four subsets of $\mathbb{C}$ are subfields of $\mathbb{C}$. For each one of these you should *prove* that it is or is not a subfield.