NAP 2019, CLASS #3, MAY 09, 2019

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• With excellent class participation, we used symmetries of the square (viewed as permutations of the vertices) to find a dihedral group of order 8 inside S_4, namely, $\{(1),(1234),(13)(24),(1432),(13),(24),(12)(34),(14)(23)\} = \langle(1234),(13)\rangle$. Pointed out that it's not normal and that there are three such subgroups of S_4, all of them conjugates.

• We explained effect of conjugation on cycles. Showed details for 3-cycles:

$$g(a\ b\ c)g^{-1} = (g(a)\ g(b)\ g(c))$$

(in Garlingspeak).

• Defined *vector space over* K.

• Defined *linear dependence* and *linear independence* for finite sets of vectors. Maybe mumbled something about an infinite set being linearly independent if and only if each finite subset is linearly indepent. Warned students that it is illegal to take infinite linear combinations, e.g., $\sum_{i=1}^{\infty} a_i \boldsymbol{v}_i$.

• Proved in detail the basic stuff on vector spaces: Theorems 1.1–1.4, Corollary to Theorem 1.2, Corollary 1. Students should read about *linear mappings* (homomorphisms between two vector spaces over the same field) and Corollary 2. Bring questions if you have any.

Comments on the homework

• Ah yes, Problem 1.16. The easiest way to prove the first assertion, for people familiar with cardinal arithmetic, is to observe that every finite-dimensional (in fact, every countable-dimensional) vector space over $\mathbb{Q}$ is countable, whereas $\mathbb{R}$ is uncountable (Cantor's "diagonal" argument). Cardinal arithmetic, however, has essentially nothing to do with the subject matter of this course (except possibly to prove, later, the existence of transcendental numbers). Therefore, what Garling probably had in mind, particularly in view of the second sentence (a question), is to find an explicit infinite set $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \dots\}$ of real numbers that is linearly independent over $\mathbb{Q}$. Remember that this means that every *finite* subset of $\{\boldsymbol{r}_i\}_{i=1}^{\infty}$ is linearly independent, equivalently, the set $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_n\}$ is linearly independent for each $n \geq 1$. Soon (see next page) we shall water down the problem, but for now we shall indicate two such lists that can be shown to be linearly independent (but to do so with what we have covered so far would be quite hard, or at least very messy.

• The first list: For each $n \geq 1$, let p_n denote the n^{th} prime number, and put $\boldsymbol{u}_n = \sqrt{p_n}$. (Thus the list is $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \dots$.)

Fact 1: The field $\mathbb{Q}(\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n)$ (that is, the smallest subfield of $\mathbb{R}$ that contains $\{\boldsymbol{u}_1, \dots, \boldsymbol{u}_n\}$) has dimension 2^n as a vector space over $\mathbb{Q}$. (The only way we know to prove this (at least without getting overwhelmed with messy computations) is to use some Galois Theory.)

• The second list: For each $n \geq 1$, let $\boldsymbol{w}_n = 2^{(2^{-n})}$, the 2^nth root of 2. (Thus the list is $\{\sqrt{2}, \sqrt[4]{2}, \sqrt[8]{2}, \dots\}$.)

Fact 2: The field $\mathbb{Q}(\boldsymbol{w}_n)$ has dimension 2^n as a vector space over $\mathbb{Q}$. (Soon you'll be able to give a very simple proof of this fact. The key ingredients are Eisenstein's Criterion (Theorem 5.2), which, with luck, we should get to next week) and Theorem 4.4, which will probably be done during Week 3.)

• Problem 1.16, watered down. This is the revised problem that we ask you to do. First of all, if you can find an infinite list *and* prove it is linearly indepent, go ahead. That will be more than enough. Otherwise, do the following:

(1) Assuming **Fact 1**, prove that $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n, \ldots\}$ is linearly independent over $\mathbb{Q}$.
(2) Assuming **Fact 2**, prove that $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n, \ldots\}$ is linearly independent over $\mathbb{Q}$.
(3) Prove (directly, without using **Fact 1**) that $\{\sqrt{2}, \sqrt{3}, \sqrt{5}\}$ is linearly independent over $\mathbb{Q}$. You may use without proof the fact that if n is a positive integer that is not the square of any other positive integer, then $\sqrt{n}$ is not a rational number.