NAP 2018 Module-V: Applications of Galois theory Problem set 1

(1) (i) Let $\alpha = \sqrt[3]{2}$ and ω be a cube root of unity. Prove that $\alpha + \omega$ is a primitive element of $\mathbb{Q}(\alpha, \omega)/\mathbb{Q}$.
 (ii) For the field extension $\mathbb{F}_8/\mathbb{F}_2$ find a primitive element.
 (iii) For a prime number p, find the number of intermediate fields for the extension $\mathbb{F}_{p^{12}}/\mathbb{F}_p$. How many of them are proper, i.e. different from $\mathbb{F}_p$ and $\mathbb{F}_{p^{12}}$?

(2) For a prime number p let $\mathbb{F}_p$ be the field with p elements. Let $E = \mathbb{F}_p(X, Y)$ and $F = \mathbb{F}_p(X^p, Y^p)$. Consider the field extension E/F. Prove that there exist infinitely many intermediate fields. (Warning: $\mathbb{F}_p$ is not algebraically closed.)

(3) Let ζ be the primitive 9-th roots of unity in $\mathbb{C}$.
 (i) Verify that $\Phi_9(X) = \Phi_3(X^3)$.
 (ii) Find the irreducible polynomial for ζ over $\mathbb{Q}$.
 (iii) Determine the Galois group of $\mathbb{Q}(\zeta)/\mathbb{Q}$.

(4) Consider the Galois extension $\mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ and the norm map N. For $a, b, c, d \in \mathbb{Q}$, find $N(a + \sqrt{2}b + \sqrt{3}c + \sqrt{6}d)$.

(5) Let $d > 0$ be a square-free integer (or rational number). For the quadratic field extension $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ verify the Hilbert's Theorem 90. In other words, for $a, b \in \mathbb{Q}$ with $N(a + b\sqrt{d}) = a^2 - db^2 = 1$ find $x, y \in \mathbb{Q}$ such that

$$a + b\sqrt{d} = \frac{x + y\sqrt{d}}{x - y\sqrt{d}}.$$