Exercise 1. Let p be a prime number and $f(X) = X^4 + pX + p \in \mathbb{Q}[X]$.

(a) Compute $R_f(X)$ and $D(f)$. Conclude that the zeros of $R_f(X)$ satisfy $Z_R \subset \{\pm 1, \pm p, \pm p^2\}$. Check that ± 1 and $\pm p^2$ are not roots of $R_f(X)$ for any p, but $R_f(p) = p^2(p-5)$ and $R_f(-p) = p^2(p-3)$. Conclude that $R_f(X)$ has a root in $\mathbb{Q}$ if and only if $p = 3, 5$.

(b) Prove that $G_f = S_4$ if $p \neq 3, 5$.

(c) If $p = 3$, prove that $G_f = D_4$.

(d) If $p = 5$, prove that $G_f = C_4$.

Exercise 2. (a) List all irreducible polynomials of degree 2 in $\mathbb{F}_5[X]$.

(b) Let $\mathbb{F}_5[X]/(X^2+3) = \mathbb{F}_5[\alpha]$, $\mathbb{F}_5[X]/(X^2+2) = \mathbb{F}_5[\beta]$ and $\mathbb{F}_5[X]/(X^2+X+1) = \mathbb{F}_5[\gamma]$. Construct explicit isomorphisms $\mathbb{F}_5[\alpha] \xrightarrow{\varphi} \mathbb{F}_5[\beta] \xrightarrow{\psi} \mathbb{F}_5[\gamma]$.

(c) Find a generator g for $\mathbb{F}_5[\alpha]^\times$, and use g and the isomorphisms φ and ψ found in (b) to produce generators of $\mathbb{F}_5[\gamma]^\times$ and $\mathbb{F}_5[\beta]^\times$.

Exercise 3. Let $f(X)$ be a polynomial of degree 6 in $\mathbb{F}_5[X]$, and let $K = \mathbb{F}_5[X]/(f)$. How many elements $\alpha \in K$ satisfy $K^\times = \langle \alpha \rangle$? How many elements $\beta \in K$ satisfy $K = \mathbb{F}_5[\beta]$?

Exercise 4. Let $K = \mathbb{F}_{3^n}$, with $n \geq 2$.

(a) How many elements have their square in K?

(b) Prove that the product P of all elements of $K^\times$ equals 2.

(c) Prove that the additive group $(K, +)$ is not cyclic.