

**Exercise 1.** Let  $f(X) \in \mathbb{Q}[X]$  be such that  $\text{Disc}(f) = 0$ . Prove that if  $\deg(f) = 3$ , then all the roots of  $f(X)$  are in  $\mathbb{Q}$ . Show with an example, that this is not necessarily true if  $f(X)$  has degree 4.

SOLUTION: Let  $f(X) = aX^3 + bX^2 + cX + d \in \mathbb{Q}[X]$ . If  $\text{Disc}(f) = 0$ , then  $f(X)$  has at least two equal roots, and  $f(X) = a \cdot (X - \alpha)^2(X - \beta)$ , with  $\alpha, \beta \in \mathbb{C}$ .

If  $\alpha = \beta$ , then

$$aX^3 + bX^2 + cX + d = a \cdot (X - \alpha)^3 = a(X^3 - 3\alpha X^2 + 3\alpha^2 X - \alpha^3).$$

Comparing coefficients we get  $\alpha = \frac{b}{3a} \in \mathbb{Q}$  and all three roots of  $f(X)$  are in  $\mathbb{Q}$ .

If  $\alpha \neq \beta$ , we have two possibilities. Either  $\alpha \in \mathbb{Q}$ , in which case  $\beta$  also in  $\mathbb{Q}$  and we are done, or  $\alpha \notin \mathbb{Q}$ . Suppose, hence, that  $\alpha \notin \mathbb{Q}$ , and let  $g(X)$  be its minimal polynomial over  $\mathbb{Q}$ , which be separable and of degree at least 2. Then both  $g(X)$  and  $g(X)^2$  must divide  $f(X)$  which is impossible since  $4 > 3$ .

We conclude that a polynomial in  $\mathbb{Q}[X]$  of degree 3 and discriminant 0 has all of its roots in  $\mathbb{Q}$ . This is not the case if  $\deg(f) = 4$ , since the polynomial  $f(X) = (X^2 + X + 1)^2$  has no root in  $\mathbb{Q}$  and has discriminant 0.  $\square$

**Exercise 2.** Let  $f(X) = X^4 + 1 \in \mathbb{Q}(X)$ .

(a) Compute the splitting field  $\mathbb{Q}_f$  and its Galois group  $G_f$ .

SOLUTION: We have  $f(X) = (X - \sqrt{i})(X - i\sqrt{i})(X + \sqrt{i})(X + i\sqrt{i})$ , with  $\sqrt{i} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$ . Consequently  $\mathbb{Q}(f) = \mathbb{Q}(\sqrt{2}, i)$ , and  $G_f = \langle \rho, \tau \rangle$ , with  $\rho(\sqrt{2}) = -\sqrt{2}$ ,  $\rho(i) = i$ , and  $\tau(\sqrt{2}) = \sqrt{2}$ ,  $\tau(i) = -i$ .

(b) Describe explicitly the immersion of  $G_f$  into  $S_4$ .

SOLUTION: Let  $\alpha_1 = \sqrt{i}$ ,  $\alpha_2 = -\sqrt{i}$ ,  $\alpha_3 = i\sqrt{i}$ ,  $\alpha_4 = -i\sqrt{i}$ . Then,

$$\rho(\sqrt{i}) = \rho\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i = -\sqrt{i}, \quad \rho(-\sqrt{i}) = \sqrt{i}$$

$$\rho(i\sqrt{i}) = \rho\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i = -i\sqrt{i}, \quad \rho(-i\sqrt{i}) = i\sqrt{i}$$

$$\tau(\sqrt{i}) = \tau\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i = -i\sqrt{i}, \quad \tau(-i\sqrt{i}) = \sqrt{i}$$

$$\tau(i\sqrt{i}) = \tau\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i = -\sqrt{i}, \quad \tau(-\sqrt{i}) = i\sqrt{i}$$

$$\begin{array}{lcl} G_f & \hookrightarrow & S_4 \\ \rho & \mapsto & (12)(34) \Rightarrow G_f \simeq V. \\ \tau & \mapsto & (13)(24) \end{array}$$

(c) Describe explicitly the Galois correspondence between the subgroups of  $G_f$  and the subfields of  $\mathbb{Q}_f$  containing  $\mathbb{Q}$ .

SOLUTION:

$$\begin{array}{lcl}
\{H < G\} & \leftrightarrow & \{\mathbb{Q} \subset M \subset \mathbb{Q}(\sqrt{2}, i)\} \\
G & \leftrightarrow & \mathbb{Q} \\
\langle \rho \rangle & \leftrightarrow & \mathbb{Q}[i] \\
\langle \tau \rangle & \leftrightarrow & \mathbb{Q}[\sqrt{2}] \\
\langle \rho\tau \rangle & \leftrightarrow & \mathbb{Q}[\sqrt{2}, i] \\
\{\text{Id.}\} & \leftrightarrow & \mathbb{Q}(\sqrt{2}, i)
\end{array}$$

□

**Exercise 3.** Let  $f(X) = X^4 - 2 \in \mathbb{Q}(X)$ .

(a) Compute its splitting field  $\mathbb{Q}_f$ , its Galois group  $G_f$  and write explicitly the Galois correspondence between the subgroups of  $G_f$  and the subfields of  $\mathbb{Q}_f$  containing  $\mathbb{Q}$ .

SOLUTION: We have  $f(X) = (X - \sqrt[4]{2})(X - i\sqrt[4]{2})(X + \sqrt[4]{2})(X + i\sqrt[4]{2})$ . Consequently  $\mathbb{Q}(f) = \mathbb{Q}(\sqrt[4]{2}, i)$ , and  $G_f = \langle \rho, \tau \rangle$ , with  $\rho(\sqrt[4]{2}) = i\sqrt[4]{2}$ ,  $\rho(i) = i$ , and  $\tau(\sqrt[4]{2}) = \sqrt[4]{2}$ ,  $\tau(i) = -i$ . We let  $\alpha_1 = \sqrt[4]{2}$ ,  $\alpha_2 = i\sqrt[4]{2}$ ,  $\alpha_3 = -\sqrt[4]{2}$ ,  $\alpha_4 = -i\sqrt[4]{2}$ . Then,

$$\rho(\alpha_1) = \alpha_2, \rho(\alpha_2) = \alpha_3, \rho(\alpha_3) = \alpha_4, \rho(\alpha_4) = \alpha_1,$$

$$\tau(\alpha_1) = \alpha_1, \tau(\alpha_2) = \alpha_4, \tau(\alpha_3) = \alpha_3, \tau(\alpha_4) = \alpha_2,$$

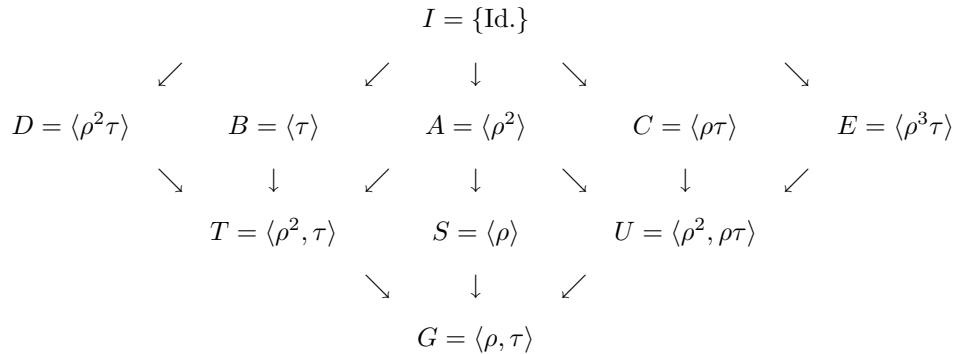
and so

$$\begin{array}{lcl}
G_f & \simeq & S_4 \\
\rho & \mapsto & (1234) \Rightarrow G_f \simeq D_4. \\
\tau & \mapsto & (23)
\end{array}$$

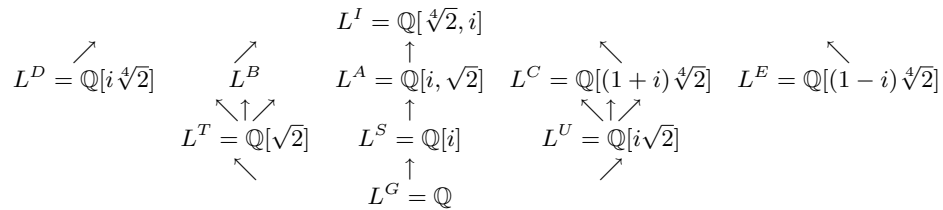
Finally, the correspondence

$$\{H < G\} \leftrightarrow \{\mathbb{Q} \subset M \subset \mathbb{Q}(\sqrt[4]{2}, i) = K\}$$

is given by the following diagrams



and



□

(b) Compute the cubic resolvent of  $f$ ,  $R(X) \in \mathbb{Q}[X]$ . Find its splitting field  $M = \mathbb{Q}_R$  and its Galois group  $G_R$ .

SOLUTION:  $R(X) = X(X^2 + 8) = (X - 2\sqrt{2}i)(X + 2\sqrt{2}i) \in \mathbb{C}[X]$  and, hence,  $M = \mathbb{Q}[i\sqrt{2}]$ . Since the minimal polynomial of  $i\sqrt{2}$  is  $X^2 + 2$ ,  $G_R = \langle \phi \rangle \simeq (12) \subset S_2$ , with  $\phi(i\sqrt{2}) = -i\sqrt{2}$ .

(c) Verify that  $M \subset \mathbb{Q}$  and compute  $\text{Gal}(\mathbb{Q}_f/M) \subset G_f$ .

SOLUTION: Clearly,  $M = \mathbb{Q}[i\sqrt{2}] \subset \mathbb{Q}[\sqrt[4]{2}, i]$ . In the tables of the Galois correspondence we see that  $\text{Gal}(\mathbb{Q}_f/M) = U = \langle \rho^2, \rho\tau \rangle$ .

(d) Compute the discriminants of  $f(X)$  and  $R(X)$ .

SOLUTION:  $\text{Disc}(f) = \text{Disc}(R) = -16 \cdot 2 \cdot 64 = -2048$ . □

**Exercise 4.** Compute the Galois group of the following cubic polynomials:

$$f(X) = X^3 - X - 1, \quad g(X) = X^3 - 3X - 1$$

SOLUTION:  $\text{Disc}(X^3 - X - 1) = -23 \Rightarrow G_f = S_3$ ;  $\text{Disc}(X^3 - 3X - 1) = 229 \Rightarrow G_g = A_3$ . □