Exercise 1. Let $f(X) \in \mathbb{Q}[X]$ be such that $\mathrm{Disc}(f) = 0$. Prove that if $\deg(f) = 3$, then all the roots of $f(X)$ are in $\mathbb{Q}$. Show with an example, that this is not necessarily true if $f(X)$ has degree 4.

SOLUTION: Let $f(X) = aX^3 + bX^2 + cX + d \in \mathbb{Q}[X]$. If $\mathrm{Disc}(f) = 0$, then $f(X)$ has at least two equal roots, and $f(X) = a \cdot (X - \alpha)^2 (X - \beta)$, with $\alpha, \beta \in \mathbb{C}$.
If $\alpha = \beta$, then

$$aX^3 + bX^2 + cX + d = a \cdot (X - \alpha)^3 = a(X^3 - 3\alpha X^2 + 3\alpha^2 X - \alpha^3).$$

Comparing coefficients we get $\alpha = \frac{b}{3a} \in \mathbb{Q}$ and all three roots of $f(X)$ are in $\mathbb{Q}$.
If $\alpha \neq \beta$, we have two possibilities. Either $\alpha \in \mathbb{Q}$, in which case β also in $\mathbb{Q}$ and we are done, or $\alpha \notin \mathbb{Q}$. Suppose, hence, that $\alpha \notin \mathbb{Q}$, and let $g(X)$ be its minimal polynomial over $\mathbb{Q}$, which be separable and of degree at least 2. Then both $g(X)$ and $g(X)^2$ must divide $f(X)$ which is impossible since $4 > 3$.

We conclude that a polynomial in $\mathbb{Q}[X]$ of degree 3 and discriminant 0 has all of its roots in $\mathbb{Q}$. This is not the case if $\deg(f) = 4$, since the polynomial $f(X) = (X^2 + X + 1)^2$ has no root in $\mathbb{Q}$ and has discriminant 0. □

Exercise 2. Let $f(X) = X^4 + 1 \in \mathbb{Q}(X)$.
(a) Compute the splitting field $\mathbb{Q}_f$ and its Galois group G_f.

SOLUTION: We have $f(X) = (X - \sqrt{i})(X - i\sqrt{i})(X + \sqrt{i})(X + i\sqrt{i})$, with $\sqrt{i} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$. Consequently $\mathbb{Q}(f) = \mathbb{Q}(\sqrt{2}, i)$, and $G_f = \langle \rho, \tau \rangle$, with $\rho(\sqrt{2}) = -\sqrt{2}$, $\rho(i) = i$, and $\tau(\sqrt{2}) = \sqrt{2}, \tau(i) = -i$.

(b) Describe explicitly the inmersion of G_f into S_4.

SOLUTION: Let $\alpha_1 = \sqrt{i}$, $\alpha_2 = -\sqrt{i}$, $\alpha_3 = i\sqrt{i}$, $\alpha_4 = -i\sqrt{i}$. Then,

$$\rho(\sqrt{i}) = \rho\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i = -\sqrt{i},\ \rho(-\sqrt{i}) = \sqrt{i}$$

$$\rho(i\sqrt{i}) = \rho\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i = -i\sqrt{i},\ \rho(-i\sqrt{i}) = i\sqrt{i}$$

$$\tau(\sqrt{i}) = \tau\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i = -i\sqrt{i},\ \tau(-i\sqrt{i}) = \sqrt{i}$$

$$\tau(i\sqrt{i}) = \tau\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i = -\sqrt{i},\ \tau(-\sqrt{i}) = i\sqrt{i}$$

$$\begin{array}{ccc} G_f & \hookrightarrow & S_4 \\ \rho & \mapsto & (12)(34) \\ \tau & \mapsto & (13)(24) \end{array} \Rightarrow G_f \simeq V.$$

(c) Describe explicitly the Galois correspondence between the subgroups of G_f and the subfields of $\mathbb{Q}_f$ containing $\mathbb{Q}$.

SOLUTION:

$$\begin{array}{ccc}
\{H < G\} & \leftrightarrow & \{\mathbb{Q} \subset M \subset \mathbb{Q}(\sqrt{2}, i)\} \\
G & \leftrightarrow & \mathbb{Q} \\
\langle \rho \rangle & \leftrightarrow & \mathbb{Q}[i] \\
\langle \tau \rangle & \leftrightarrow & \mathbb{Q}[\sqrt{2}] \\
\langle \rho\tau \rangle & \leftrightarrow & \mathbb{Q}[\sqrt{2}, i] \\
\{\text{Id.}\} & \leftrightarrow & \mathbb{Q}(\sqrt{2}, i)
\end{array}$$

□

Exercise 3. Let $f(X) = X^4 - 2 \in \mathbb{Q}(X)$.

(a) Compute its splitting field $\mathbb{Q}_f$, is Galois group G_f and write explicitly the Galois correspondence between the subgroups of G_f and the subfields of $\mathbb{Q}_f$ containing $\mathbb{Q}$.

SOLUTION: We have $f(X) = (X - \sqrt[4]{2})(X - i\sqrt[4]{2})(X + \sqrt[4]{2})(X + i\sqrt[4]{2})$. Consequently $\mathbb{Q}(f) = \mathbb{Q}(\sqrt[4]{2}, i)$, and $G_f = \langle \rho, \tau \rangle$, with $\rho(\sqrt[4]{2}) = i\sqrt[4]{2}$, $\rho(i) = i$, and $\tau(\sqrt[4]{2}) = \sqrt[4]{2}, \tau(i) = -i$. We let $\alpha_1 = \sqrt[4]{2}$, $\alpha_2 = i\sqrt[4]{2}$, $\alpha_3 = -\sqrt[4]{2}$, $\alpha_4 = -i\sqrt[4]{2}$. Then,

$$\rho(\alpha_1) = \alpha_2,\ \rho(\alpha_2) = \alpha_3,\ \rho(\alpha_3) = \alpha_4,\ \rho(\alpha_4 = \alpha_1,$$

$$\tau(\alpha_1) = \alpha_1,\ \tau(\alpha_2) = \alpha_4,\ \tau(\alpha_3) = \alpha_3,\ \tau(\alpha_4 = \alpha,$$

and so

$$\begin{array}{ccc}
G_f & \hookrightarrow & S_4 \\
\rho & \mapsto & (1234) \\
\tau & \mapsto & (23)
\end{array} \Rightarrow G_f \simeq D_4.$$

Finally, the correspondence

$$\{H < G\} \leftrightarrow \{\mathbb{Q} \subset M \subset \mathbb{Q}(\sqrt[4]{2}, i) = K\}$$

is given by the following diagrams

$$\begin{array}{ccccccccc}
 & & & & I = \{\text{Id.}\} & & & & \\
 & \swarrow & & \swarrow & \downarrow & \searrow & & \searrow & \\
D = \langle \rho^2\tau \rangle & & B = \langle \tau \rangle & & A = \langle \rho^2 \rangle & & C = \langle \rho\tau \rangle & & E = \langle \rho^3\tau \rangle \\
 & \searrow & \downarrow & \swarrow & \downarrow & \searrow & \downarrow & \swarrow & \\
 & & T = \langle \rho^2, \tau \rangle & & S = \langle \rho \rangle & & U = \langle \rho^2, \rho\tau \rangle & & \\
 & & & \searrow & \downarrow & \swarrow & & & \\
 & & & & G = \langle \rho, \tau \rangle & & & &
\end{array}$$

and

$$\begin{array}{ccccc}
 & & L^I = \mathbb{Q}[\sqrt[4]{2}, i] & & \\
\nearrow & \nearrow & \uparrow & \nwarrow & \nwarrow \\
L^D = \mathbb{Q}[i\sqrt[4]{2}] & L^B & L^A = \mathbb{Q}[i, \sqrt{2}] & L^C = \mathbb{Q}[(1+i)\sqrt[4]{2}] & L^E = \mathbb{Q}[(1-i)\sqrt[4]{2}] \\
 & \nwarrow\uparrow\nearrow & \uparrow & \nwarrow\uparrow\nearrow & \\
 & L^T = \mathbb{Q}[\sqrt{2}] & L^S = \mathbb{Q}[i] & L^U = \mathbb{Q}[i\sqrt{2}] & \\
 & \nwarrow & \uparrow & \nearrow & \\
 & & L^G = \mathbb{Q} & &
\end{array}$$

□

(b) Compute the cubic resolvent of f, $R(X) \in \mathbb{Q}[X]$. Find its splitting field $M = \mathbb{Q}_R$ and its Galois group G_R.

SOLUTION: $R(X) = X(X^2+8) = (X-2\sqrt{2}i)(X+2\sqrt{2}i) \in \mathbb{C}[X]$ and, hence, $M = \mathbb{Q}[i\sqrt{2}]$. Since the minimal polinomial of $i\sqrt{2}$ is X^2+2, $G_R = \langle \phi \rangle \simeq (12) \subset S_2$, with $\phi(i\sqrt{2}) = -i\sqrt{2})$.

(c) Verify that $M \subset \mathbb{Q}$ and compute $\mathrm{Gal}(\mathbb{Q}_f/M) \subset G_f$.

SOLUTION: Clearly, $M = \mathbb{Q}[i\sqrt{2}] \subset \mathbb{Q}[\sqrt[4]{2}, i]$. In the tables of the Galois correspondence we see that $\mathrm{Gal}(\mathbb{Q}_f/M) = U = \langle \rho^2, \rho\tau \rangle$.

(d) Compute the discriminants of $f(X)$ and $R(X)$.

SOLUTION: $\mathrm{Disc}(f) = \mathrm{Disc}(R) = -16 \cdot 2 \cdot 64 = -2048$. □

Exercise 4. Compute the Galois group of the following cubic polynomials:

$$f(X) = X^3 - X - 1, \; g(X) = X^3 - 3X - 1$$

SOLUTION: $\mathrm{Disc}(X^3 - X - 1) = -23 \Rightarrow G_f = S_3; \mathrm{Disc}(X^3 - 3X - 1) = 229 \Rightarrow G_g = A_3$.

□