Exercise 1. Let $f(X) \in \mathbb{Q}[X]$ be such that $\mathrm{Disc}(f) = 0$. Prove that if $\deg(f) = 3$, then all the roots of $f(X)$ are in $\mathbb{Q}$. Show with an example, that this is not necessarily true if $f(X)$ has degree 4.

Exercise 2. Let $f(X) = X^4 + 1 \in \mathbb{Q}(X)$.

(a) Compute the splitting field $\mathbb{Q}_f$ and its Galois group G_f.

(b) Describe explicitly the inmersion of G_f into S_4.

(c) Describe explicitly the Galois correspondence between the subgroups of G_f and the subfields of $\mathbb{Q}_f$ containing $\mathbb{Q}$.

Exercise 3. Let $f(X) = X^4 - 2 \in \mathbb{Q}(X)$.

(a) Compute its splitting field $\mathbb{Q}_f$, its Galois group G_f in S_4 and write explicitly the Galois correspondence between the subgroups of G_f and the subfields of $\mathbb{Q}_f$ containing $\mathbb{Q}$.

(b) Compute the cubic resolvent of f, $R(X) \in \mathbb{Q}[X]$. Find its splitting field $\mathbb{Q}_R = M$ and its Galois group G_M in S_3.

(c) Verify that $M \subset \mathbb{Q}$ and compute $\mathrm{Gal}(\mathbb{Q}_f/M) \subset G_f$.

(d) Compute the discriminants of $f(X)$ and $R(X)$.

Exercise 4. Compute the Galois group of the following cubic polynomials in $\mathbb{Q}[X]$:

(a) $f(X) = X^3 - X - 1$.

(b) $g(X) = X^3 - 3X - 1$.