1. Let p be a prime. Show that $X^p - X \in \mathbf{F}_p[X]$ is separable, while $X^p - 1 \in \mathbf{F}_p[X]$ is not. How many zeroes do these polynomials have in $\mathbf{F}_p$?

2. Let $n \geq 3$.
   (a) Let α be a zero of $X^n - 1 \in \mathbf{Q}[X]$. Show that $\mathbf{Q}(\alpha)$ is a normal extension of $\mathbf{Q}$.
   (b) Let β be a zero of $X^n - 2 \in \mathbf{Q}[X]$. Show that $\mathbf{Q}(\beta)$ is not a normal extension of $\mathbf{Q}$.

3. Let $\zeta_9 \in \mathbf{C}$ denote a primitive 9-th root of unity and let $F = \mathbf{Q}(\zeta_9)$.
   (a) Show that the factorization into irreducible factors of $x^9 - 1$ over $\mathbf{Q}$ is given by

$$(x^9 - 1) = (x - 1)(x^2 + x + 1)(x^6 + x^3 + 1).$$

   (b) Show that $\mathrm{Aut}_{\mathbf{Q}}(F)$ is isomorphic to $\mathbf{Z}/9\mathbf{Z}^* \cong \mathbf{Z}/6\mathbf{Z}$.
   (c) Exhibit $\gamma \in F$ so that $[\mathbf{Q}(\gamma) : \mathbf{Q}] = 3$.

4. Let R be the ring $\mathbf{Q}[X]/(X^4 + X + 1)$. For a polynomial $g(X) \in \mathbf{Q}[X]$, we write $\overline{g(X)}$ for its canonical image in R.
   (a) Show that every element of R can be represented by a polynomial in $\mathbf{Q}[X]$ of degree ≤ 3.
   (b) Let $g(X) = X^2+1$. Compute $\overline{g(X)}^2$ and represent the result by a polynomial in $\mathbf{Q}[X]$ of degree ≤ 3.