1. Determine the degree of the splitting field of the polynomial $f = X^4 - 2$ over the following fields:
$$\mathbf{C}, \quad \mathbf{R}, \quad \mathbf{Q}, \quad \mathbf{Q}(\sqrt{2}), \quad \mathbf{Q}(\sqrt[4]{2}).$$

2. (a) Show that there exist trascendental elements $\alpha, \beta \in \mathbf{C}$ such that their product is algebraic over $\mathbf{Q}$.
   (b) Show that there exist trascendental elements $\alpha, \beta \in \mathbf{C}$ such that both their sum is algebraic over $\mathbf{Q}$.
   (c) Let $\alpha, \beta \in \mathbf{C}$ have the property that both their sum and their product are algebraic over $\mathbf{Q}$. Show that α and β themselves are algebraic over $\mathbf{Q}$.

3. Let $\zeta_8 = e^{\frac{2\pi i}{8}} \in \mathbf{C}$.
   (a) Show that ζ_8 is a primitive 8-th root of unity and determine its minimum polynomial over $\mathbf{Q}$.
   (b) Show that $\mathbf{Q}(i) \subset \mathbf{Q}(\zeta_8)$ and that $\mathbf{Q}(\sqrt{2}) \subset \mathbf{Q}(\zeta_8)$.
   (c) How many elements do the following sets have?
$$\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}(\sqrt{2}), \mathbf{C}), \quad \mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}(\zeta_8), \mathbf{C}), \quad \mathrm{Hom}_{\mathbf{Q}(\sqrt{2})}(\mathbf{Q}(\zeta_8), \mathbf{C})$$

4. Let p be a prime and let $\mathbf{F}$ be a field of characteristic p.
   (a) Show that $\mathbf{F}^p = \{x^p : x \in \mathbf{F}\}$ is a subfield of $\mathbf{F}$.
   (b) When $\mathbf{F} = \mathbf{Z}/p\mathbf{Z}(T)$ is the field of rational functions in the variable T, compute $[\mathbf{F} : \mathbf{F}^p]$.