NEPAL ALGEBRA PROJECT 2018
MODULE 1 — HOMEWORK #1:
WEDNESDAY, 09 MAY, 2018

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1. Factor X^4+1 into irreducible factors
   (a) in $\mathbb{Q}[X]$;
   (b) in $\mathbb{R}[X]$;
   (c) in $\mathbb{C}[X]$;
   (d) In $\mathbb{F}_2[X]$. ($\mathbb{F}_2$ is the two-element field $\mathbb{Z}/2\mathbb{Z}$.)

In each case, justify that your factors are irreducible.

2. Find all automorphisms of each of the following fields:
   (a) $\mathbb{Q}[\sqrt{-1}] = \{a+b\sqrt{-1} \mid a,b \in \mathbb{Q}\}$;
   (b) $\mathbb{Q}(\sqrt{2}) = \{a+b\sqrt{2} \mid a,b \in \mathbb{Q}\}$;
   (c) $\mathbb{Q}[\sqrt[3]{2}] = \{a+b\sqrt[3]{2}+c\sqrt[3]{4} \mid a,b,c \in \mathbb{Q}\}$;
   (d) $\mathbb{Q}[\sqrt[4]{2}] = \{a+b\sqrt[4]{2}+c\sqrt{2}+d\sqrt{2}\sqrt[4]{2} \mid a,b,c,d \in \mathbb{Q}\}$.

(An automorphism of a field F is an isomorphism $F \to F$.) In each ease, justify that the map you describe is an automorphism, and explain why these are the only automorphisms.)

3. Use the Euclidean algorithm to find the GCD $h(X)$, in $\mathbf{F}_3[X]$, of $f(X) = X^4+X^3+X^2+2$ and $g(X) = X^3+X+2$, and write $h(X)$ as a linear combination of $f(X)$ and $g(X)$ with coefficients in $\mathbf{F}_3[X]$. ($\mathbf{F}_3$ is the three-element field $\mathbb{Z}/3\mathbb{Z}$.)

4. Let $f : R \to S$ be a homomorphism of rings.
   (a) If J is an ideal of S, prove that $f^{-1}(J) := \{r \in R \mid f(r) \in J\}$ is an ideal of R.
   (b) Give an example of a ring homomorphism $f : R \to S$ and an ideal I of R such that $f(I) := \{f(r) \mid r \in I\}$ is not an ideal of S.

5. Prove that if p is a prime number then $p \mid \binom{p}{\ell}$ for all ℓ with $0 < \ell < p$. Deduce that the "Freshman's Dream", $(f(X)+g(X))^p = f(X)^p + g(X)^p$, holds in $\mathbb{F}_p[X]$.

These problems are due Tuesday, 15 May, 2018, at 10 pm Nepal time. They must be sent to nap@rnta.eu by 10 pm Nepal time (with copies to rwiegand1@unl.edu and swiegand1@unl.edu. You may discuss problems with other students in the class, but you must do the write-up completely by yourself, without consulting anyone else. You may refer, by number, to theorems, propositions, etc., in Milne's book, up through Remark 1.18, and also to methods and results presented in class during the first week (08 May — 10 May). If you are ambitious, you can write your solutions in TeX and send them as an attachment. Alternatively, you can write them out (legibly, please), scan them, and send as an attachment. A less desirable option would be to photograph your solutions and send the photo; this will probably be harder to read, so the first two options are preferable.

You can download Milne's book at http://www.jmilne.org/math/CourseNotes/FT.pdf
The NAP website is: http://www.rnta.eu/nap/

Feel free to email us anytime with questions.