NEPAL ALGEBRA PROJECT 2018
MODULE 1 — HOMEWORK #0 (PREPARATORY, NOT COUNTED FOR CREDIT): SOLUTIONS

1. Let G be a group in which $x^2 = 1$ for all x. Prove that G is abelian.

Solution: For $x, y \in G$, we have $1 = (xy)^2 = xyxy$. Multiplying this equation by x on the left and y on the right, we get $xy = x1y = xxyxyy = x^2yxy^2 = 1yx1 = yx$.

2. Let R be a ring in which $x^2 = x$ for all x. Prove that $2x = 0$ for all x and that R is commutative.

Solution: For each $x, y \in R$ one has $x + y = (x + y)^2 = x^2 + xy + yx + y^2 = x + xy + yx + y$. Adding $-x - y$ to both sides, we get $0 = xy + yx$. Therefore

$$(**) \qquad xy = -yx \quad \text{for all } x, y \in R\,.$$

Taking $y = x$ in (**), we have $x = x^2 = -x^2 = -x$:

$$(***) \qquad x = -x \quad \text{for all } x \in R\,.$$

Now (***) says that $2x = 0$, and (**) and (***) together say that $xy = yx$.

Gratuitous remarks: A ring in which $x^2 = x$ for all x is called a Boolean ring. For an example, take any set X and let R be the collection of subsets of X. For $A, B \in R$, let $A + B = (A \cup B) \setminus (A \cap B)$, and let $A \cdot B = A \cap B$. For a more interesting example, let X be any compact, Hausdorff, totally disconnected topological space, and let R be the collection of clopen (closed and open) subsets of X, with the two operations above. The Stone Representation Theorem says that *every* Boolean ring arises in this way.

3. Factor $f(X) = X^3 + 2X - 1$ into irreducible factors (a) over $\mathbb{Q}$, (b) over $\mathbb{C}$, (c) over $\mathbf{F}_5$; and justify that your factors are irreducible.

Solution: (a) Since it's a cubic we just have to test for rational roots. Proposition 1.11 (which, admittedly, we did not really discuss until Thursday) says that we need only check whether or not 1 or -1 is a root. Since $f(1) = 2$ and $f(-1) = -4$, $f(X)$ has no roots in $\mathbb{Q}$ and hence is irreducible in $\mathbb{Q}[X]$.

(b) Over $\mathbb{C}$, every non-constant polynomial factors into linear factors. ("How was I supposed to know that?" you say. Good point!) Anyway, you will learn this fact later in the course. But that still does not say what the three roots are. From looking at the graph, one can see that $f(X)$ has exactly one real root, and then one can say, using **GP2**, that the other two roots are complex conjugates. Thus the factorization looks like this: $f(X) = (X - \alpha)(X - \beta)(X - \overline{\beta})$, where α is real (but irrational), β is a non-real complex number, and $\overline{\beta}$ is its complex conjugate. That's about all one can say without using the (very complicated) formula for the roots of a cubic. If you are interested, take a look at https://math.vanderbilt.edu/schectex/courses/cubic/ . There is an even more horrible formula for roots of a quartic, but not for polynomials of degree five or higher. The non-existence of such formulas for degree five and above is one of the really beautiful results you will learn later in this course.

(c) As in (a), it's enough to test for roots. And again, this is a finite problem, but for a very different reason, namely, that $\mathbf{F}_5$ has only 5 elements. Rather than listing them as $0, 1, 2, 3, 4$, it's sometimes easier use $0, \pm 1, \pm 2$. Just plug 'em in and see if the result is congruent to 0 modulo 5. We have $f(0) = -1$, $f(1) = 2$, $f(-1) = -4$, $f(2) = 11$, and $f(-2) = -13$, none of them a multiple of 5. Therefore $f(X)$ is irreducible.