Examples: A generator of $\mathbb{F}_7^\times$ is 3. If $K = \mathbb{F}_3(X)/(X^2+1) = \mathbb{F}_3[\alpha]$, a generator of $K^\times$ will be, for example, $g = 1+\alpha$.

Theorem. Fix a prime p.

(a) For every power $q = p^n$, there is a field $\mathbb{F}_q$ with q elements.

(b) Every field with $q = p^n$ elements is the splitting field of $X^q - X$ over $\mathbb{F}_p$. Therefore, all finite fields with q elements are isomorphic. Notation: $\mathbb{F}_q$.

(c) Let $f(x) \in \mathbb{F}_p[X]$ be an irreducible polynomial of degree n. Let α be a zero of f. Then,

(i) $\alpha^{p^n} = \alpha$;

(ii) n is the smallest possible integer for which (a) holds;

(iii) $f(X) = (X-\alpha)(X-\alpha^p)\cdots(X-\alpha^{p^{n-1}})$.

(d) If K is a subfield of $\mathbb{F}_{p^n}$, then the cardinality of K is p^d for some divisor d of n.

(e) For every divisor d of n there is a unique subfield of $\mathbb{F}_{p^n}$ of cardinality p^d.

Example. Both polynomials X^2+1 and X^2+2X+2 are irreducible in $\mathbb{F}_3[X]$. Thus, they provide us with two ways of obtaining fields of cardinality 9, both isomorphic to $\mathbb{F}_9$ and to each other:

$$\mathbb{F}_3(X)/(X^2+1) = \mathbb{F}_3[\alpha] = \{0,1,2,\alpha,1+\alpha,2+\alpha,2\alpha,1+2\alpha,2+2\alpha | \alpha^2+1=0 \Leftrightarrow \alpha^2 = 2\}$$

$$\mathbb{F}_3(X)/(X^2+2X+2) = \mathbb{F}_3[\beta] = \{0,1,2,\beta,1+\beta,2+\beta,2\beta,1+2\beta,2+2\beta | \beta^2+2\beta+2=0 \Leftrightarrow \beta^2 = \beta+1\}$$

Let us construct an explicit isomorphism φ bewteen $\mathbb{F}_3[\alpha]$ and $\mathbb{F}_3[\beta]$. Since $\alpha^2 = 2$, necessarily $\varphi(\alpha)^2 = 2 \in \mathbb{F}_3[\beta]$. Thus, candidates for $\varphi(\alpha)$ are elements $a+b\beta \in \mathbb{F}_5[\beta]$ with $(a+b\beta)^2 = 2$. We look for them.

$$(a+b\beta)^2 = 2 \Leftrightarrow a^2 + 2ab\beta + b^2\beta^2 = 2 \Leftrightarrow a^2 + 2ab\beta + b^2\beta + b^2 = 2$$

$$\Leftrightarrow \begin{cases} a^2+b^2 = 2 \\ b(2a+b) = 0 \end{cases} \Leftrightarrow \begin{cases} b = a, a^2+b^2 = 2 \Rightarrow a = \pm 1, b = a \\ b = 0, a^2 = 2, \text{impossible} \end{cases}$$

We thus have two options for isomorphisms φ betweenF$_3[\alpha]$ and $\mathbb{F}_3[\beta]$:

$$\alpha \mapsto 1+\beta, \quad \text{or } \alpha \mapsto -1-\beta.$$

Also, the generator $g = 1+\alpha$ of $\mathbb{F}_3[\alpha]^\times$ maps, respectively, to to $2+\beta$ and $-\beta$, both generators of $\mathbb{F}_3[\beta]^\times$.