NAProject 2018 Module IV: Computing Galois Groups Lecture 4: Thursday, June 26, 2018

Elements in S_3 and parity:

Number of elements permuted		parity
0	Id.	even
2	$(12), (13), (23)$	odd
3	$(123), (132)$	even

Thus, $A_3 = \{\text{Id.}, (123), (132) = (123)^2\}$, of size 3, prime, so it has no proper subgrups.
Transitive subgroups of S_3 and size:

Order of the subgroup	
6	S_3
3	A_3

Elements in S_4 and parity:

Number of elements permuted		parity
0	Id.	even
2	$(12), (13), (14), (23), (24), (34)$	odd
3	$(123), (132), (124), (142), (134), (143), (234), (243)$	even
4	$(1234), (1432), (1342), (1243), (1324), (1423)$	odd
4	$(13)(24), (12)(34), (14)(32)$	even

Thus, $A_4 = \langle (123), (124), (134), (234), (13)(24), (12)(34) \rangle$ of size 12.
Subgroups of A_4 and size:

Order of the subgroup	
1	Id.
4	$\{\text{Id.}, (13)(24), (12)(34), (14)(32)\} = \langle (13)(24), (12)(34) \rangle = V$
3	$\{\text{Id.}, (123), (132)\} = \langle (123) \rangle \sim C_3$
	$\{\text{Id.}, (124), (142)\} = \langle (124) \rangle \sim C_3$
	$\{\text{Id.}, (134), (143)\} = \langle (134) \rangle \sim C_3$
	$\{\text{Id.}, (234), (243)\} = \langle (234) \rangle \sim C_3$

Transitive subgroups of S_4 and size divisible by 4:

Order of the subgroup	
4	$\langle (1234) \rangle \sim C_4$
	$\langle (1243) \rangle \sim C_4$
	$\langle (1324) \rangle \sim C_4$
	$\langle (13)(24), (12)(34) \rangle = V$
8	$\langle (1234), (13) \rangle \sim D_4$
	$\langle (1324), (12) \rangle \sim D_4$
	$\langle (1243), (14) \rangle \sim D_4$

Proposition 3: Let $\text{char} K \neq 2$ and $f(X) \in K[X]$ be a separable polynomial of degree n. Let $\sigma \in G_f$. Then $G_f \subset A_n \Leftrightarrow \Delta(f) \in K \Leftrightarrow \text{Disc}(f)$ is a square in K.

Galois groups of irreducible separable polynomials of degree 3:

degree of $f(X)$	$\text{Disc}(f)$ in K	G_f
3	$\square$	A_3
3	$\neq \square$	S_3

Examples:

$$h(X) = X^3 - 3X + 1, \text{Disc}(h) = -4(-3)^2 - 27 = 81 = 9^2 \in \mathbb{Q}^2 \Rightarrow G_h = A_3$$

$$g(X) = X^3 + 3X + 1, \text{Disc}(g) = -135 \notin \mathbb{Q}^2 \Rightarrow G_h = S_3$$