Example 1. $Aut_{\mathbf{Q}}(\mathbf{Q}(\sqrt{2})) \cong \mathbf{Z}/2\mathbf{Z}$.

Sol.: An automorphism $\phi \in \mathrm{Aut}_{\mathbf{Q}}(\mathbf{Q}(\sqrt{2}))$ is completely determined once we fix the image of $\sqrt{2}$. The choice $\phi(\sqrt{2}) = \sqrt{2}$ determines the identity element; the choice $\phi(\sqrt{2}) = -\sqrt{2}$ determines the homomorphism

$$\phi\colon \mathbf{Q}(\sqrt{2}) \to \mathbf{Q}(\sqrt{2}), \quad \phi(m + \sqrt{2}n) = m - \sqrt{2}n. \tag{1}$$

Since $\mathbf{Q}(\sqrt{2})$ is a finite degree extension of $\mathbf{Q}$, a $\mathbf{Q}$-homomorphism is also an isomorphism: it is injective because it is a field homomorphism; it is surjective because it is an injective $\mathbf{Q}$-linear map of a finite dimensional vector space into itself.

Excercise.
(a) Verify that ϕ defined in (1) is a field homomorphism:

$$\phi((a + \sqrt{2}b)(c + \sqrt{2}d)) = \phi((a + \sqrt{2}b))\phi((c + \sqrt{2}d)), \quad \forall a, b, c, d \in \mathbf{Q}$$

(b) Verify that $\phi \circ \phi = Id$.

Example 2. $Aut_{\mathbf{R}}(\mathbf{C}) \cong \mathbf{Z}/2\mathbf{Z}$.

Sol.: An automorphism $\phi \in \mathrm{Aut}_{\mathbf{R}}(\mathbf{C})$ is completely determined once we fix the image of i. The choice $\phi(i) = i$ determines the identity element; the choice $\phi(i) = -i$ determines the homomorphism

$$\phi\colon \mathbf{C} \to \mathbf{C}, \quad \phi(x + iy) = x - iy, \tag{2}$$

which is the usual complex conjugation $z \mapsto \bar{z}$, $z = x + iy$. Since $\mathbf{C}$ is a finite degree extension of $\mathbf{R}$, then ϕ is an isomorphism.

Excercise.
(a) Verify that ϕ defined in (2) is a field homomorphism:

$$\phi((a + ib)(c + id)) = \phi((a + ib))\phi((c + id)), \quad \forall a, b, c, d \in \mathbf{R}$$

(b) Verify that $\phi \circ \phi = Id$.

Example 3. *Let ξ be a primitive 5^{th}-root of 1. Then $Aut_{\mathbf{Q}}(\mathbf{Q}(\xi)) \cong \mathbf{Z}/5\mathbf{Z}^* \cong \mathbf{Z}/4\mathbf{Z}$.*

Sol.: As $\mathbf{Q}(\xi)$ is a finite extension of $\mathbf{Q}$, we have

$$\mathrm{Aut}_{\mathbf{Q}}(\mathbf{Q}(\xi)) = \mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}(\xi), \mathbf{Q}(\xi)),$$

and the above set is in 1-1 correspondence with

$$Z(x^4 + x^3 + x^2 + x + 1) \cap \mathbf{Q}(\xi) = \{\xi,\ \xi^2,\ \xi^3,\ \xi^4\},$$

the zeroes of $x^4 + x^3 + x^2 + x + 1$ contained in $\mathbf{Q}(\xi)$. Hence there are four automorphisms in $\mathrm{Aut}_{\mathbf{Q}}(\mathbf{Q}(\xi))$, each determined by the image of ξ:

$$\phi_i(\xi) = \xi^i, \quad i = 1, \ldots, 4.$$

This determines an isomorphism $\mathrm{Aut}_{\mathbf{Q}}(\mathbf{Q}(\xi)) \to \mathbf{Z}/5\mathbf{Z}^*$, $\phi_i \mapsto i$.

Remark. Note that $\mathbf{Q}(\xi)$ is the splitting field of the polynomial $f(x) = x^4 + x^3 + x^2 + x + 1$. So, in principle, $\mathrm{Aut}_{\mathbf{Q}}(\mathbf{Q}(\xi))$ is a subgroup of S_4, the permutation group of 4 elements. In this case, the zeroes of f are very special, and $\mathrm{Aut}_{\mathbf{Q}}(\mathbf{Q}(\xi))$ is a much smaller subgroup than S_4.

Remark. In general, let ξ be a primitive n^{th}-root of 1. Then $\mathrm{Aut}_{\mathbf{Q}}(\mathbf{Q}(\xi)) \cong \mathbf{Z}/n\mathbf{Z}^*$.

Excercise. Fix $\xi = e^{2\pi i/5}$. Verify that the automorphism determined by the condition $\phi(\xi) = \xi^2$ has order 4, that is $\phi \circ \phi \circ \phi \circ \phi = Id$.