Example 1. *Compute the cardinality of* $\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}(\sqrt[4]{2}), \mathbf{F})$, *where* $\mathbf{F} = \mathbf{Q}, \mathbf{R}, \mathbf{C}$.

Sol.: The minimal polynomial of $\sqrt[4]{2}$ over $\mathbf{Q}$ is $f(x) = x^4 - 2$.
Its zeroes are $Z(f) = \{\sqrt[4]{2}, -\sqrt[4]{2}, \sqrt[4]{2}i, -\sqrt[4]{2}i\}$.

(a) $\#\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}(\sqrt[4]{2}), \mathbf{Q}) = 0$.

Since f has no zeroes in $\mathbf{Q}$, there are no $\mathbf{Q}$-homomorphisms $\phi\colon \mathbf{Q}(\sqrt[4]{2}) \to \mathbf{Q}$.
Another way to argue: since $\dim_{\mathbf{Q}} \mathbf{Q}(\sqrt[4]{2}) = 4$, $\dim_{\mathbf{Q}} \mathbf{Q} = 1$ and every field homomorphism is necessarily injective, there are no $\mathbf{Q}$-homomorphisms $\phi\colon \mathbf{Q}(\sqrt[4]{2}) \to \mathbf{Q}$.

(b) $\#\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}(\sqrt[4]{2}), \mathbf{R}) = 2$.

The polynomial f has 2 zeroes in $\mathbf{R}$, namely $\sqrt[4]{2}, -\sqrt[4]{2}$. Hence there are two $\mathbf{Q}$-homomorphisms $\phi_1, \phi_2\colon \mathbf{Q}(\sqrt[4]{2}) \to \mathbf{R}$, determined by
$\phi_1(\sqrt[4]{2}) = \sqrt[4]{2}$ and $\phi_2(\sqrt[4]{2}) = -\sqrt[4]{2}$, respectively.

(c) $\#\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}(\sqrt[4]{2}), \mathbf{C}) = 4$.

The polynomial f has 4 zeroes in $\mathbf{C}$. Hence there are four $\mathbf{Q}$-homomorphisms $\phi_1, \phi_2, \phi_3, \phi_4\colon \mathbf{Q}(\sqrt[4]{2}) \to \mathbf{C}$, determined by
$\phi_1(\sqrt[4]{2}) = \sqrt[4]{2}, \quad \phi_2(\sqrt[4]{2}) = -\sqrt[4]{2},$
$\phi_3(\sqrt[4]{2}) = \sqrt[4]{2}i, \quad \phi_4(\sqrt[4]{2}) = -\sqrt[4]{2}i$, respectively.

Example 2. *Compute the cardinality of* $\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}(\sqrt{2}, \sqrt{3}), \mathbf{F})$, *where* $\mathbf{F} = \mathbf{Q}, \mathbf{R}, \mathbf{C}$.

Sol.:
(a) $\#\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}(\sqrt{2}, \sqrt{3}), \mathbf{Q}) = 0$.
Consider the subfield $\mathbf{Q}(\sqrt{2}) \subset \mathbf{Q}(\sqrt{2}, \sqrt{3})$. Since the minimal polynomial $x^2 - 2$ of $\sqrt{2}$ over $\mathbf{Q}$ has no roots in $\mathbf{Q}$, there are no $\mathbf{Q}$-homomorphisms $\phi\colon \mathbf{Q}(\sqrt{2}) \to \mathbf{Q}$. It follows that there are no $\mathbf{Q}$-homomorphisms $\phi\colon \mathbf{Q}(\sqrt{2}, \sqrt{3}) \to \mathbf{Q}$.

(b) $\#\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}(\sqrt{2}, \sqrt{3}), \mathbf{R}) = 4$.
There are two $\mathbf{Q}$-homomorphisms $\phi_1, \phi_2\colon \mathbf{Q}(\sqrt{2}) \to \mathbf{Q}$, determined by
$\phi_1(\sqrt{2}) = \sqrt{2}$ and $\phi_2(\sqrt{2}) = -\sqrt{2}$, respectively.
Now we extend them to $\mathbf{Q}(\sqrt{2}, \sqrt{3})$, by viewing $\mathbf{Q}(\sqrt{2}, \sqrt{3}) = \mathbf{Q}(\sqrt{2})(\sqrt{3})$. The polynomial $x^2 - 3$ is irreducible in $\mathbf{Q}(\sqrt{2})[x]$, hence it is the minimal polynomial of $\sqrt{3}$ over $\mathbf{Q}(\sqrt{2})$. Its zeroes $\pm\sqrt{3} \in \mathbf{R}$. Hence $\#\mathrm{Hom}_{\mathbf{Q}(\sqrt{2})}(\mathbf{Q}(\sqrt{2}, \sqrt{3}), \mathbf{R}) = 2$. This means that given a $\mathbf{Q}$-homomorphism $\mathbf{Q}(\sqrt{2}) \to \mathbf{R}$, we can extend it to $\mathbf{Q}(\sqrt{2}, \sqrt{3})$ in two ways, depending on whether $\sqrt{3}$ is mapped to $\sqrt{3}$ or $-\sqrt{3}$.
The four $\mathbf{Q}$-homomorphisms $\phi_1, \phi_2, \phi_3, \phi_4\colon \mathbf{Q}(\sqrt{2}, \sqrt{3}) \to \mathbf{R}$ are determined by
$\phi_1(\sqrt{2}) = \sqrt{2}, \quad \phi_1(\sqrt{3}) = \sqrt{3},$
$\phi_2(\sqrt{2}) = -\sqrt{2}, \quad \phi_2(\sqrt{3}) = \sqrt{3},$
$\phi_3(\sqrt{2}) = \sqrt{2}, \quad \phi_3(\sqrt{3}) = -\sqrt{3},$
$\phi_4(\sqrt{2}) = -\sqrt{2}, \quad \phi_4(\sqrt{3}) = -\sqrt{3}$, respectively.

(c) $\#\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}(\sqrt{2}, \sqrt{3}), \mathbf{C}) = 4$.
As above.

Example 3. *Compute the cardinality of* $\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}(\xi), \mathbf{F})$, *where* $\xi = e^{2\pi i/5}$ *is a primitive* 5^{th} *root of 1 and* $\mathbf{F} = \mathbf{R}, \mathbf{C}$.

Sol.: Observe that over $\mathbf{Q}$ the polynomial $x^5 - 1$ decomposes as $x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$, with $x^4 + x^3 + x^2 + x + 1$ irreducible (see Milne, Lemma

1.41). This shows that $x^4+x^3+x^2+x+1$ is the minimal polynomial of ξ over $\mathbf{Q}$. Its roots in $\mathbf{C}$ are ξ,ξ^2,ξ^3,ξ^4, none of which is real. One has $\xi^4=\bar{\xi}$ and $\xi^3=\bar{\xi}^2$.

(a) from the above discussion, it follows that $\#\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}(\xi),\mathbf{R})=0$.

(b) $\#\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}(\xi),\mathbf{C})=4$
There four homomorphisms $\phi_1,\phi_2,\phi_3,\phi_4\colon \mathbf{Q}(\xi)\to\mathbf{C}$, determined by $\phi_1(\xi)=\xi$, $\phi_2(\xi)=\xi^2,\quad \phi_3(\xi)=\xi^3,\quad \phi_4(\xi)=\xi^4$, respectively.

Example 4. *Compute the cardinality of* $\mathrm{Hom}_{\mathbf{Q}(\sqrt{5})}(\mathbf{Q}(\xi),\mathbf{C})$*, where* $\xi=e^{2\pi i/5}$ *is a primitive* 5^{th} *root of 1.*

Sol.: Observe that $\mathbf{Q}(\sqrt{5})$ is a subfield of $\mathbf{Q}(\xi)$: for this it is sufficient to check that $\sqrt{5}=2(\xi+\xi^4)=2(\xi+\bar{\xi})\in\mathbf{Q}(\sqrt{5})$. Over $\mathbf{Q}(\sqrt{5})$, the polynomial $x^4+x^3+x^2+x+1$ decomposes into two degree 2 irreducible factors

$$(x^2-(\frac{-1+\sqrt{5}}{2})x+1)(x^2+(\frac{1+\sqrt{5}}{2})x+1),$$

where $(x^2-(\frac{-1+\sqrt{5}}{2})x+1)=(x-\xi)(x-\bar{\xi})$ and $(x^2+(\frac{1+\sqrt{5}}{2})x+1)=(x-\xi^2)(x-\bar{\xi}^2)$. As a subfield of $\mathbf{Q}(\xi)$, the field $\mathbf{Q}(\sqrt{5})$ is obtained by adjoining $\xi+\bar{\xi}=(\frac{-1+\sqrt{5}}{2})$ or equivalently $\xi^2+\bar{\xi}^2=-(\frac{1+\sqrt{5}}{2})$ to $\mathbf{Q}$:

$$\mathbf{Q}\subset\mathbf{Q}(\sqrt{5})=\mathbf{Q}(\xi+\bar{\xi})=\mathbf{Q}(\xi^2+\bar{\xi}^2)\subset\mathbf{Q}(\xi)$$

For an automorphism in $\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}(\xi),\mathbf{C})$ to fix $\mathbf{Q}(\sqrt{5})$ it is necessary and sufficient that either $\xi\mapsto\xi$ or $\xi\mapsto\bar{\xi}$.
Conclusion: $\#\mathrm{Hom}_{\mathbf{Q}(\sqrt{5})}(\mathbf{Q}(\xi),\mathbf{C})=2$.