Example 1. *Let* $\mathbf{F}$ *be a field and let* $\mathbf{F} \subset \mathbf{E}$ *be a field extension. Let* α *and* β *be algebraic elements in* $\mathbf{E}$*, with minimal polynomial over* $\mathbf{F}$ *of degree* n *and* m *respectively. Then* $[\mathbf{F}(\alpha,\beta):\mathbf{F}] \leqslant n \cdot m$*.*

Sol.: The simple extension $\mathbf{F} \subset \mathbf{F}(\alpha)$ has degree $[\mathbf{F}(\alpha):\mathbf{F}] = n$. The degree of the simple extension $\mathbf{F}(\alpha) \subset \mathbf{F}(\alpha,\beta)$ is $\leqslant m$: this is because the minimal polynomial of β, which is irreducible over $\mathbf{F}$, could become reducible over $\mathbf{F}(\alpha)$. Hence the minimal polynomial of β over $\mathbf{F}(\alpha)$ has degree $\leqslant m$ (we can say no more).

By the multiplicativity of the degrees we obtain

$$[\mathbf{F}(\alpha,\beta):\mathbf{F}] = [\mathbf{F}(\alpha):\mathbf{F}][[\mathbf{F}(\alpha,\beta):\mathbf{F}(\alpha)] \leqslant n \cdot m.$$

Example 2. *Compute the degree* $[\mathbf{Q}(\sqrt{5},\sqrt{11}):\mathbf{Q}]$.

Sol. Over $\mathbf{Q}$, the minimal polynomial of $\sqrt{5}$ is $f(x) = x^2 - 5$ and the minimal polynomial of $\sqrt{11}$ is $g(x) = x^2 - 11$. To see that g remains irreducible over $\mathbf{Q}(\sqrt{5})$ it is sufficient to check that $\sqrt{11} \notin \mathbf{Q}(\sqrt{5})$:
if $\sqrt{11} \in \mathbf{Q}(\sqrt{5})$, then it can be written as $\sqrt{11} = a + b\sqrt{5}$, with $a, b \in \mathbf{Q}$. Squaring both terms, we get

$$11 = a^2 + 5b^2 + 2ab\sqrt{5} \quad \Leftrightarrow \quad \begin{cases} a^2 + 5b^2 = 11 \\ 2ab\sqrt{5} = 0 \end{cases} .$$

The above system has no solution in $\mathbf{Q}$.
Conclusion: $[\mathbf{Q}(\sqrt{5},\sqrt{11}):\mathbf{Q}] = 2 \cdot 2 = 4$

Example 3. *Compute the degree* $[\mathbf{Q}(\sqrt{2},\sqrt[4]{2}):\mathbf{Q}]$.

Sol.: Over $\mathbf{Q}$, the minimal polynomial of $\sqrt{2}$ is $f(x) = x^2 - 2$ and the minimal polynomial of $\sqrt[4]{2}$ is $g(x) = x^4 - 2$.
Let's show that in this case $[\mathbf{Q}(\sqrt{2},\sqrt[4]{2}):\mathbf{Q}] < 2 \cdot 4 = 8$:
since $\sqrt{2} = (\sqrt[4]{2})^2$ we have that $\mathbf{Q}(\sqrt{2},\sqrt[4]{2}) = \mathbf{Q}(\sqrt[4]{2})$ and $[\mathbf{Q}(\sqrt{2},\sqrt[4]{2}):\mathbf{Q}] = 4 < 8$.

Example 4. *Compute the degree* $[\mathbf{Q}(\sqrt[3]{2},\omega):\mathbf{Q}]$*, where* $\omega = e^{2\pi i/3}$.

Sol.: In a previous excercise (see Lecture1-examples, page 1, n.7) we saw that $\mathbf{Q}(\sqrt[3]{2},\omega) = \mathbf{Q}(\sqrt[3]{2},\omega\sqrt[3]{2})$. Over $\mathbf{Q}$, both $\sqrt[3]{2}$ and $\omega\sqrt[3]{2}$ have the same minimal polynomial $f(x) = x^3 - 2$, of degree 3.
Let's show that in this case $[\mathbf{Q}(\sqrt[3]{2},\omega\sqrt[3]{2}):\mathbf{Q}] = 3 \cdot 2 = 6 < 9$:
we have $[\mathbf{Q}(\sqrt[3]{2}):\mathbf{Q}] = 3$. But over $\mathbf{Q}(\sqrt[3]{2})$, the polynomial f splits as

$$x^3 - 2 = (x - \sqrt[3]{2})(x^2 + \sqrt[3]{2}x + (\sqrt[3]{2})^2),$$

and the degree-2 factor $x^2 + \sqrt[3]{2}x + (\sqrt[3]{2})^2$ is the minimal polynomial of $\omega\sqrt[3]{2}$ over $\mathbf{Q}(\sqrt[3]{2})$.
Conclusion: $[\mathbf{Q}(\sqrt[3]{2},\omega\sqrt[3]{2}):\mathbf{Q}] = 3 \cdot 2 = 6$.