Proof of items (a), (b), (d) of the following proposition. Item (c) will be addressed later.

Proposition. (Milne, Prop.2.4) *For every polynomial* $f \in \mathbf{F}[x]$*, there exists a field extension* $\mathbf{F} \subset \mathbf{K}$ *with the property that*
(a) f*, as a polynomial in* $\mathbf{K}[x]$*, decomposes as* $f(x) = c(x - \alpha_1) \ldots (x - \alpha_m)$*, with* $\alpha_i \in \mathbf{K}, c \in \mathbf{K}$*;*
(b) $\mathbf{F}(\alpha_1, \ldots, \alpha_m) = \mathbf{K}$.

In addition,
(c) the field $\mathbf{K}$ *is unique up to* $\mathbf{F}$*-isomorphism;*
(d) If $\deg(f) = n$*, then* $[\mathbf{K} : \mathbf{F}] \leq n!$.

Definition. An algebraically closed field is a field $\mathbf{F}$ with the property that every polynomial $f \in \mathbf{F}[x]$ splits as $f(x) = c(x - \alpha_1) \ldots (x - \alpha_m)$, with $\alpha_i, c \in \mathbf{F}$.

In other words, "all zeroes of every polynomial $f \in \mathbf{F}[x]$ are in $\mathbf{F}$".

Example. The field of complex numbers $\mathbf{C}$ is algebraically closed: by the Fundamental of Algebra, every non-constant polynomial in $\mathbf{C}[x]$ has a zero in $\mathbf{C}$. This implies that every $f \in \mathbf{C}[x]$ splits as $f(x) = c(x - \alpha_1) \ldots (x - \alpha_m)$ in $\mathbf{C}[x]$.

Example. The fields $\mathbf{Q}$ and $\mathbf{R}$ are not algebraically closed: for example the polynomial $f(x) = x^2 + 1$ has no zero in $\mathbf{Q}$, nor in $\mathbf{R}$.

Equivalent characterizations of an algebraically closed field (Milne, Prop.1.4.2 & Def. 1.4.3(a)).

Definition. An algebraic closure of a field $\mathbf{F}$ is an extension $\overline{\mathbf{F}}$ of $\mathbf{F}$ which is algebraically closed over $\mathbf{F}$ and consists of algebraic elements over $\mathbf{F}$.

Fact. (Milne Chapter 6) For every field $\mathbf{F}$ there exists an extension $\mathbf{F} \subset \mathbf{L}$ which is algebraically closed.

Proposition. (Milne, Prop. 1.45) *Every field* $\mathbf{F}$ *admits an algebraic closure* $\overline{\mathbf{F}}$.

$\overline{\mathbf{F}}$ is unique, but *only* up to isomorphism and the isomorphism is generally not unique. Colloquially we often speak of "the" algebraic closure of a given field, but it is not correct.

Let $\mathbf{L}$ be an algebraically closed extension of $\mathbf{F}$. Then one defines

$$\overline{\mathbf{F}} = \{x \in \Omega \mid x \text{ is algebraic over } \mathbf{F}\}.$$

The proof that $\overline{\mathbf{F}}$ satisfies the required properties, i.e. is a field and it is algebraically closed, is based on the following two facts:

(1) $\mathbf{F} \subset \mathbf{K}$ fields, $\alpha \in \mathbf{K}$. Then α is algebraic over $\mathbf{F}$ if and only if there exists a finite extension $\mathbf{E}$ of $\mathbf{F}$ such that $\mathbf{F} \subset \mathbf{E} \subset \mathbf{K}$.

(2) $\mathbf{F} \subset \mathbf{K} \subset \mathbf{L}$ fields. If $\mathbf{K}$ is algebraic over $\mathbf{F}$ and $\mathbf{L}$ is algebraic over $\mathbf{K}$, then $\mathbf{L}$ is algebraic over $\mathbf{F}$.

Example. $\overline{\mathbf{R}} \cong \mathbf{C}$;

In $\mathbf{C}$ there is an algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$: it consists of the complex numbers which are algebraic over $\mathbf{Q}$. It is a proper subset of $\mathbf{C}$ because it is countable: for example it does not contain π.