FAMILY NAME .... *GIVEN NAME* .... *STUDENT ID* ....

Solve the maximum number of the problems (briefly and explaining your answers). *Write your answers in the appropriate spaces. NO ADDED SHEETS WILL BE ACCEPTED.* 1 Esercise = 5 points. Exam length: 2 hours. No question allowed during the first hour and during the last 20 minutes.

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1.
   (a) Find the minimal polynomial of $2 \cdot 2^{1/3} + 3$ over $\mathbf{Q}$, and prove that it is the minimal polynomial.

   (b) Prove that $\mathbf{Q}(2 \cdot 2^{1/3} + 2) = \mathbf{Q}(2^{1/3})$ and that $\mathbf{Q}(2^{1/3}) \neq \mathbf{Q}(\sqrt{2})$

2. Let R be a domain (i.e. a commutative ring without zero divisors) and suppose that F is a field contained in R (as a subring). Prove that if $\dim_F R$ is finite then R is a field. Show that the condition that $\dim_F R < \infty$ is necessary.

3. Prove the theorem about transitivity of algebraic extensions: If $F \subseteq K \subseteq L$ are field extensions such that K is algebraic over F and L is algebraic over K, then L is algebraic over F.

4. Describe all elements of the Galois group of the polynomial $x^3 - 3 \in \mathbf{Q}[x]$.

5. Give the definition of contructible number and determine which among $2^{1/3}$, $8^{1/4}$ and $\sqrt{3}+\sqrt{11}$ is constructible.

6. State in full generality the fundamental correspondence Theorem of Galois Theory.

7. Given a finite field $\mathbf{F}_q$ ($q = p^n$), consider $\gamma \in \mathbf{F}_q^*$ and let $f_\gamma(X) \in \mathbf{F}_p[X]$ be its minimal polynomial over $\mathbf{F}_p$.
   a) Show that if $m = \deg f_\gamma$, then $\gamma, \gamma^p, \gamma^{p^2}, \ldots, \gamma^{p^{m-1}}$ are exactly all the root of $f_\gamma(X)$.
   b) Show that if γ is a generator of the multiplicative group $\mathbf{F}_q^*$, then all the root of f_γ are also generators.

8.
   a) Show that for any rational number q, the real number $\cos(q\pi)$ is algebraic. *Hint: consider* $e^{i\pi q}$.
   b) Determine the minimal polynomial of $\cos(\pi/5)$.