2017 NAP Lecture Module III, Problem 4

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To solve the following problems, we assume all arguments in Chapter 1 and Chapter 2.

Problem III -4] = Excersise Chap. 3-2.

Let p be an odd prime, and set $\zeta = e^{2\pi i/p}$. Consider the field $E = \boldsymbol{Q}[\zeta]$. It is the splitting field of $f(X) = X^{p-1} + X^{p-2} + \cdots + 1$. So (by Thm. 3.10) $E/\boldsymbol{Q}$ is Galois.

(1) Show that $G = \mathrm{Gal}(E/\boldsymbol{Q})$ is isomorphic to (the cyclic group) $(\boldsymbol{Z}/(p\boldsymbol{Z}))^{\times} \cong \boldsymbol{Z}/(p-1)\boldsymbol{Z}$.

(2) Let H be the subgroup of quadratic residues in $(\boldsymbol{Z}/(p\boldsymbol{Z}))^{\times}$. Set $\alpha = \sum_{i \in H} \zeta^i, \beta = \sum_{i \in G-H} \zeta^i$.

Show that

(a) α and β are invariant under the action of H.

(b) $\sigma\alpha = \beta, \sigma\beta = \alpha$ for any $\sigma \in G - H$.

(c) $X^2 + X + \alpha\beta \in$ Q[X].

(3) By calculating $\alpha\beta$, show that

$$E^H = \begin{cases} \boldsymbol{Q}[\sqrt{p}] & p \equiv 1 (\mathrm{mod}\ 4) \\ \boldsymbol{Q}[\sqrt{-p}] & p \equiv 3 (\mathrm{mod}\ 4) \end{cases} .$$

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Given Name Family Name Date: d /m /2017

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