2017 NAP Lecture Module III, Problem 3 Solution

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To solve the following problems, we assume all arguments up to Prop. 3.20 and those in Module III.

Problem 3] (From Chapter Exercise 1-4).
(1) Let $a \in \boldsymbol{R}^+$ be a given number. Construct $\sqrt{a}$ (> 0).

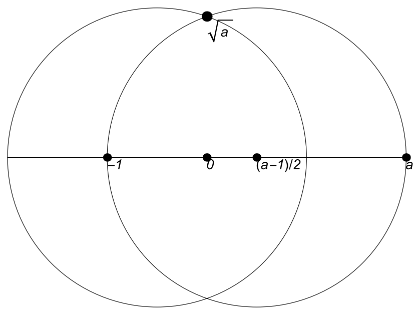

Fig. 1.1: Construction of $\sqrt{a}$

(2) We assume that we can describe the trisector of any given constructible angle. Interprete this assumption to the constructibility of the solution of the cubic equation $4x^3 - 3x - k = 0$ for any given constructible number k.

We put the constructible angle by $\cos(3\theta)$. Then (according to the addition formula) the trisection is given by $\cos(3\theta) = 4(\cos\theta)^3 - 3\cos\theta$. So, the problem is reduced to solve the equation $4x^3 - 3x = k$.

(3) Under the above assumption, show that we can construct the regular 7-gon.

To describe the regular 7-gon, it is equivalent to construct the complex number $\zeta^7 = 1$. It holds $\zeta^7 - 1 = (\zeta - 1)(\zeta^6 + \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1)$.

So, the problem is reduced to solve the equation $\zeta^3 + \zeta^{-3} + \zeta^2 + \zeta^{-2} + \zeta + \zeta^{-1} + 1 = 0$. By putting $t = \zeta + \zeta^{-1}$, we have

$$f(t) := t^3 + t^2 - 2t - 1 = \zeta^3 + \zeta^{-3} + \zeta^2 + \zeta^{-2} + \zeta + \zeta^{-1} + 1.$$

Putting $t = (2\sqrt{7})s/3 - 1/3$, we have $g(s) = \dfrac{27}{14\sqrt{7}} f(\dfrac{2\sqrt{7}}{3}s - \dfrac{1}{3}) = -\dfrac{1}{2\sqrt{7}} - 3s + 4s^3$. By assumption, we can construct the root s of $g(s) = 0$. So we can construct also ζ, under our "false" assumption.