2017 NAP Lecture Module III, Problem 2, Solution

May 30,'17, H. Shiga

We suppose all the results in the text up to Prop. 3.20..

Let $f(X) \in \boldsymbol{Q}[X]$ be an irreducible cubic polynomial, and let α be a real root of $f(X) = 0$. Set $E = \boldsymbol{Q}[\alpha]$.

Letting L be the splitting field of $f(X)$, we have only following two possibilities:

(a) $E/\boldsymbol{Q}$ is Galois ($\Longleftrightarrow L = E \Longleftrightarrow \sharp\mathrm{Gal}(E/\boldsymbol{Q}) = 3$),
(b) $E/\boldsymbol{Q}$ is not Galois ($\Longleftrightarrow E \subset_{\neq} L \Longleftrightarrow \sharp\mathrm{Gal}(L/\boldsymbol{Q}) = 6$).

<u>Problem 2-1.</u> Show the following statements.

1) If $f(X)$ has a complex (non real) root, $\boldsymbol{Q}[\alpha]$ is not Galois over F.

[Solution] $f(X)$ is the minimal polynomial of α. If $\boldsymbol{Q}[\alpha]$ is Galois, it must be the splitting field of $f(X)$. But, $\boldsymbol{Q}[\alpha]$ is contained in $\boldsymbol{R}$, it is a contradiction.

2) Under the condition 1), let L be the splitting field of f. Then $\mathrm{Gal}(L/F) \cong \mathfrak{S}_3$.

[Solution] Because $\boldsymbol{Q}[\alpha]$ is a proper subfield of L, we have $3 = [\boldsymbol{Q}[\alpha] : \boldsymbol{Q}] < [L : \boldsymbol{Q}]$. On the otherhand $\mathrm{Gal}(L/F)$ is a subgroup of $\mathfrak{S}_3$. So we must have $\mathrm{Gal}(L/F) \cong \mathfrak{S}_3$.

3) In case of (b), the discriminant δ of $f(X)$ should not be a square in $\boldsymbol{Q}$.

[Solution] Set $\alpha = \alpha_1 . \alpha_2, \alpha_3$ be the roots of $f(X)$. Let us consider the subgroup $\mathfrak{A}_3 \subset \mathfrak{S}_3 = \mathrm{Gal}(L/F)$. Corresponding to this group, we have a Galois extension $M/\boldsymbol{Q}$ of degree 2. Here M is the fixed field of $\mathfrak{A}_3$. So $M = \boldsymbol{Q}[\Delta], \Delta = (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)$. Hence Δ should not belong to $\boldsymbol{Q}$. Consequently, $\delta = \Delta^2$ is not a square in $\boldsymbol{Q}$.

<u>Problem 2-2</u>.

1) Let $f(X) = X^3 - aX - b \in \boldsymbol{Q}[X]$ be an irreducible cubic polynomial. Let α be a real root of $f(X)$. Show a (necessary and sufficient) condition that $E/\boldsymbol{Q}, (E = \boldsymbol{Q}[\alpha])$ becomes to be Galois.

[Solution] Let us show " $E/\boldsymbol{Q}$:Galois $\Longleftrightarrow \delta \in \boldsymbol{Q}^2$ ". In Problem 2-1 (3) we showed "$\Longleftarrow$" , so it is enough to see $\Longrightarrow$.

Suppose $E/\boldsymbol{Q}$:Galois. Because $[E : \boldsymbol{Q}] = 3$, $\mathrm{Gal}(E/\boldsymbol{Q})$ is a subgroup of $\mathfrak{S}_3$ of index 2. It must be $\mathfrak{A}_3$. $\Delta = (\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)$ is invariant under the action of $\mathfrak{A}_3$. It means $\Delta \in \boldsymbol{Q}$. Consequently δ should be a square in $\boldsymbol{Q}$.

2) For $f(X) = X^3 - 3X - b$, $(b \in \boldsymbol{Z})$, find all cases so that we have Galois $E/\boldsymbol{Q}$.

(By the way, for $f(X) = X^3 - 3X - b, (b \in \boldsymbol{Q})$, the problem is a bit challenging. How about the case $f(X) = X^3 - aX - 1, (a \in \boldsymbol{Q})$? Very challenging)

NAP Lecture Part III Problem 2 Sheet

Given Name   Family Name   Date: d   /m   /2017

Status   Specialized Field

[Solutions]