2017 NAP Lecture Module III, Problem 2

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We suppose all the results in the text up to Prop. 3.20..

Let $f(X) \in \boldsymbol{Q}[X]$ be an irreducible cubic polynomial, and let α be a real root of $f(X) = 0$. Set $E = \boldsymbol{Q}[\alpha]$.

Letting L be the splitting field of $f(X)$, we have only following two possibilities:

(a) $E/\boldsymbol{Q}$ is Galois ($\Longleftrightarrow L = E \Longleftrightarrow \sharp\mathrm{Gal}(E/\boldsymbol{Q}) = 3$),
(b) $E/\boldsymbol{Q}$ is not Galois ($\Longleftrightarrow E \subset_{\neq} L \Longleftrightarrow \sharp\mathrm{Gal}(L/\boldsymbol{Q}) = 6$).

Problem 2-1. Show the following statements.
1) If $f(X)$ has a complex (non real) root, $f[\alpha]$ is not Galois over F.
2) Under the condition 1), let L be the splitting field of f. Then $\mathrm{Gal}(L/F) \cong \mathfrak{S}_3$.
3) In case of (b), the discriminant δ of $f(X)$ should not be a square in $\boldsymbol{Q}$.

Problem 2-2.
1) Let $f(X) = X^3 - aX - b \in \boldsymbol{Q}[X]$ be an irreducible cubic polynomial. Let α be a real root of $f(X)$. Show a (necessary and sufficient) condition that $E/\boldsymbol{Q}$ becomes to be Galois.
2) For $f(X) = X^3 - 3X - b$, $(b \in \boldsymbol{Z})$, find all cases so that we have Galois $E/\boldsymbol{Q}$.
(By the way, for $f(X) = X^3 - aX - b, (1 \leq a \leq 9)$, we have only 6 Galois $E/\boldsymbol{Q}$'s.)

NAP Lecture Part III Problem 2 Sheet

Given Name Family Name Date: d /m /2017

Status Specialized Field

[Solutions]