NEPAL ALGEBRA PROJECT 2017
MODULE 2 — HOMEWORK #2:
TUESDAY, 23 MAY, 2017

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1. Let F be a finite field. Prove that $|F|$ is a power of p.

2. Let $f(X) = X^5 - 3$, and let $\zeta = e^{\frac{2\pi i}{5}}$.
   (a) Show that ζ is a root of $g(X) = X^4 + X^3 + X^2 + X + 1$.
   (b) Prove that $g(X)$ is irreducible in $\mathbb{Q}[X]$. (Hint: First prove that $g(X + 1)$ is irreducible.)
   (c) Show that $E = \mathbb{Q}(\sqrt[5]{3}), \zeta)$ is the splitting field of $f(X)$ over $\mathbb{Q}$.
   (d) Describe $\mathrm{Aut}(E/\mathbb{Q}(\zeta))$.
   (e) Describe $\mathrm{Aut}(E/\mathbb{Q}(\sqrt[5]{3}))$.
   (f) What is $|\operatorname{Aut} E|$?

3. Let $\alpha \in \mathbb{C}$ be a root of $f(X) = X^3 - 3X - 1$. Prove that $\sqrt{2} \notin \mathbb{Q}(\alpha)$.

4. Let $\alpha = \sqrt{3 + 2\sqrt{2}} \in \mathbb{C}$.
   (a) Find the minimal polynomial $f(X)$ of α over $\mathbb{Q}$.
   (b) Find the splitting field E of $f(X)$ over $\mathbb{Q}$.
   (c) Describe the group $\mathrm{Aut}(E/\mathbb{Q})$.

5. Describe the field $\mathbb{F}_2[X]/(X^2 + X + 1)$ by giving
   (a) a list of elements,
   (b) the addition table, and
   (c) the multiplication table.

6. Find the splitting field E of $f(X) = X^4 + X^2 + 1$ over $\mathbb{Q}$, and describe the group $\mathrm{Aut}(E/\mathbb{Q})$.

7. (This is essentially Milne Exercise 4-4 (page 57).) Find an extension $E/\mathbb{Q}$ of degree 4 such that there does not exist a proper intermediate field $\mathbb{Q} \subset F \subset E$. For this problem you may use the following results (which are true, by the way) without proving them:
   (A) There exists a polynomial $f(X) \in \mathbb{Q}[X]$ of degree 4 whose splitting field has degree 24 over $\mathbb{Q}$.
   (B) The group A_4 consisting of even permutations of $\{1, 2, 3, 4\}$ is the only subgroup of order 12 in S_4.

8. Milne Exercise 4-9 (page 57)

These problems are due Tuesday, 30 May, 2017, at 10 pm Nepal time. They must be sent to nap@rnta.eu (copy to rwiegand1unl.edu and swiegand1unl.edu) by 10 pm Nepal time. You may discuss problems with other students in the class, but you must do the write-up completely by yourself, without consulting anyone else. You may refer, by number, to theorems, propositions, etc., in Milne's book, provided that they are results that were covered in NAP Module 1 or Module 2: up to 25 May). If you are ambitious, you can write your solutions in TeX and send them as an attachment. Alternatively, you can write them out (legibly, please), scan them, and send as an attachment. A less desirable option would be to photograph your solutions and send the photo; this will probably be harder to read, so the first two options are preferable.

You can download Milne's book at http://www.jmilne.org/math/CourseNotes/FT.pdf The NAP website is: http://www.rnta.eu/nap/ Feel free to email us anytime with questions.