Nepal Algebra Project(NAP)
Central Department of Mathematics
Tribhuvan University, Kirtipur, Kathmandu,Nepal
Fields and Galois Theory- Short Note of Module 4 - Lecture 4
Course Instructor: René Schoof and Laura Geatti

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The Galois group of a polynomial

$\mathbf{K}$ a field, $f \in \mathbf{K}[x]$ a separable monic polynomial, $\mathbf{K}_f = \mathbf{K}[\alpha_1, \ldots, \alpha_n]$ the splitting field of f, $\{\alpha_i\}_i$ zeros of f, $G_f := Gal(\mathbf{K}_f/\mathbf{K}) \subset S_n$ the Galois group of f.

Criterion. Assume $char(\mathbf{K}) \neq 2$. Then $G_f \subset A_n \Leftrightarrow Disc(f)$ is a square in $\mathbf{K}$.

Consequence. $f \in \mathbf{K}[x]$, separable monic polynomial of *degree 3*, irreducible over $\mathbf{K}$. Then $G_f = A_3$, if $Disc(f)$ is a square in $\mathbf{K}$, and $G_f = S_3$, if $Disc(f)$ is not a square in $\mathbf{K}$.

The degree 4 case.
$f \in \mathbf{K}[x]$, separable monic polynomial of *degree* 4.

(a) If f is divisible by a degree 1 factor, we are back in the degree 3 case.

(b) If f factors as the product of two irreducible polynomials g, h of degree 2, then either $\mathbf{K}_g = \mathbf{K}_h = \mathbf{K}_f$ and $G_f \cong C_2$, or $\mathbf{K}_g \neq \mathbf{K}_h$, $\mathbf{K}_f = \mathbf{K}_g\mathbf{K}_h$ and $G_f \cong C_2 \times C_2$.

(c) If f is irreducible, then G_f is isomorphic to a transitive subgroup of S_4. These are S_4, A_4, D_4, V_4, C_4.

The group S_4 and its transitive subgroups in detail.

The *cubic resolvent* g of f: its zeros α, β, γ are fixed by $G_f \cap V_4$, hence $g \in \mathbf{K}[x]$.

Excercises. Computation of the Galois group of the following polynomials:

1. $x^2 - 101$;
2. $x^3 - 5x^2 + 6$;
3. $x^3 - 5x - 5$;
4. $x^3 - 3x + 1$;
5. $x^3 - 1$;
6. $x^3 - 3$;
7. $x^3 - 2x^2 + 3x + 5$;