Nepal Algebra Project 2016 Midterm exam

Tribhuvan University

June 25^{th} 2016

1. (a) Find the minimal polynomial of $\sqrt{3}+\sqrt{5}$ over $\mathbb{Q}$, and *prove* that it is the minimal polynomial.

   (5 marks)

   (b) Prove that $\mathbb{Q}(\sqrt{3}+\sqrt{5}) = \mathbb{Q}(\sqrt{3}, \sqrt{5})$.

   (5 marks)

2. Prove the theorem about transitivity of algebraic extensions: If $F \subseteq K \subseteq L$ are field extensions such that K is algebraic over F, and L is algebraic over K, then L is algebraic over F.

   (10 marks)

3. Let F be a finite field with $\mathrm{char}(F) = p(> 0)$. Show that $F = \{\text{roots of the equation } X^{p^n} - X = 0\}$, where $n = [F : \mathbb{F}_p]$. (*Hint.* We can use the fact that the multiplicative group $F^* = F - \{0\}$ of F is a cyclic group.See Exrcise 1-3 (e))

   (10 marks)

4. Let F be a field of characteristic $p(> 0)$. Suppose $a \in F$ is not a p-th power in F (i.e. We don't have $a = \alpha^p$ for any $\alpha \in F$). Show that $f(X) = X^p - a$ is irreducible in $F[X]$. (This is the fact of Example 2.11 stated without proof.)

   (10 marks)

5. Let $\zeta = e^{2\pi i/5}$.

   (a) Prove that $\mathbb{Q}[\zeta]$ is a Galois extension of $\mathbb{Q}$.

   (2 marks)

   (b) Calculate $|\mathbb{Q}[\zeta] : \mathbb{Q}|$.

   (2 marks)

   (c) What is the structure of the Galois group $\mathrm{Gal}(\mathbb{Q}[\zeta]/\mathbb{Q})$?

   (4 marks)

   (d) Give an example of a field M such that $\mathbb{Q} \subset M \subset \mathbb{Q}[\zeta]$.

   (2 marks)

6. (a) Prove that $X^n - 2$ is irreducible for all positive integers n.

   (2 marks)

   (b) Let $\omega = \sqrt[n]{2}$ for some positive integer n. Calculate

   $$|\mathbb{Q}[\omega] : \mathbb{Q}|.$$

   (1 mark)

   (c) Prove that $\sqrt[n]{2}$ is a constructible number if and only if $n = 2^k$ for some positive integer k.

   (7 marks)