Nepal Algebra Project(NAP)
Central Department of Mathematics
Tribhuvan University,Kirtipur, Kathmandu,Nepal
Fields and Galois Theory

Course Instructor: Prof. Michel Waldschmidt

NAP: Module-4, Problem Set 2

1. *Galois group of a family of cubic polynomials.*
For $a \in \mathbb{Z}$, consider the polynomial

$$f_a(X) = X^3 - aX^2 - (a+3)X - 1 \in \mathbb{Z}[X].$$

Further, define $\sigma : \mathbb{C} \setminus \{-1\} \to \mathbb{C} \setminus \{0\}$ by

$$\sigma : z \mapsto -\frac{1}{1+z}$$

(a) Is f is irreducible over $\mathbb{Q}$?
(b) Let α be a complex root of f_a. Check that $\sigma(\alpha)$ is also a root of f_a.
(c) What is the Galois group of f_a over $\mathbb{Q}$?
(d) Is the discriminant of f_a a square in $\mathbb{Q}$?

2. *Galois group of a polynomial of degree* 4.
Let b be an integer. Define $f(X) = X^4 + bX^2 + 1 \in \mathbb{Z}[X]$.
(a) Let α be a root of f in $\mathbb{C}$. Write the 4 roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of f in terms of α.
Deduce

$$b = \alpha^2 + \frac{1}{\alpha^2}\cdot$$

(b) For which values of b is the polynomial f separable?
For these values, what is the Galois group of f over $\mathbb{Q}$.
`Hint`. One may consider 5 cases:
- $b = -2$;
- $b = 2$;
- $-b - 2$ is a nonzero square;
- $b - 2$ is a nonzero square;
- otherwise.

http://www.rnta.eu/nap/index.php