Nepal Algebra Project 2016

Tribhuvan University

Module 4 — Problem Set 1 (MW)

1. *The symmetric group $\mathfrak{S}_4$.*

(a) Check that, among the 24 elements of the symetric group $\mathfrak{S}_4$,

- 1 has order 1
- 9 have order 2
- 8 have order 3
- 6 have order 4

`Hint:` the partitions of 4 are (1)(1)(1)(1), (2)(1)(1), (2)(2), (3)(1), (4).

(b) Deduce that in $\mathfrak{S}_4$ there are 30 subgroups:

- 1 with order 1
- 9 with order 2
- 4 with order 3
- 7 with order 4
- 4 with order 6
- 3 with order 8
- 1 with order 12
- 1 with order 24

(c) Check that there are 11 conjugacy classes and 4 normal subgroups.

`Reference:`

`http://groupprops.subwiki.org/wiki/Subgroup_structure_of_symmetric_group:S4`

2. *The dihedral group D_n of order $2n$.*

(a) Let $n \geq 1$. Consider the following two elements r and s in $\mathfrak{S}_n$:

$$r(i) = i + 1 \mod n, \quad s(i) = n + 2 - i \mod n \qquad (i = 1, 2, \ldots, n).$$

Check that $r^n = 1$, $s^2 = 1$, $rsrs = 1$ and that the subgroup D_n of $\mathfrak{S}_n$ generated by r and s has order $2n$.
This is the dihedral group of index n, group of symmetries of the regular n–gone.

(b) Show that if n is odd, then D_{2n} is isomorphic to the direct product $C_2 \times D_n$.

(c) Give the list of groups of order $2p$ with p prime.

`Hint.` Use the fact that such a group contains an element of order p and an element of order 2.

3. *A transitive subgroup of $\mathfrak{S}_n$ containing a $n-1$ cycle and a transposition is $\mathfrak{S}_n$.*

(a) Let $\sigma = (1, 2, \cdots, n-1)$ and $\tau = (1, n)$. Check

$$\sigma\tau\sigma^{-1} = (2, n).$$

(b) Check that $\mathfrak{S}_n$ is generated by τ and σ.

(c) Let G be a transitive subgroup of $\mathfrak{S}_n$ containing a $n-1$ cycle and a transposition. Check $G = \mathfrak{S}_n$.

Correction

1. *The symmetric group $\mathfrak{S}_4$.*

(a)

• There is 1 element of order 1, namely I, corresponding to the partition (1)(1)(1)(1).

• There are 9 elements of order 2, among which 6 correspond to the partition (2)(1)(1), namely the transpositions

$$(a, b),\ (a, c),\ (a, d),\ (b, c),\ (b, d),\ (c, d)$$

while the 3 others correspond to the partition (2)(2), namely the products of disjoint transpositions

$$(a, b)(c, d),\ (a, c)(b, d),\ (a, d)(b, c).$$

• There are 8 elements of order 3, belonging to 4 cyclic groups, they correspond to the partition (3)(1), they are the cycles of length 3:

$$(a, b, c),\ (a, b, d),\ (a, c, b),\ (a, c, d),\ (a, d, b),\ (a, d, c),\ (b, c, d),\ (b, d, c).$$

• There are 6 elements of order 4, corresponding to the partition (4), they are the cycles of length 4:

$$(a, b, c, d),\ (a, b, d, c),\ (a, c, b, d),\ (a, c, d, b),\ (a, d, b, c),\ (a, d, c, b).$$

(b) As a consequence, there are

• 1 subgroup of order 1, and 1 conjugacy class;

• 9 subgroups of order 2, they are cyclic, and 1 conjugacy class;

• 4 subgroups of order 3, they are cyclic, and 1 conjugacy class;

• 7 subgroups of order 4, and 3 conjugacy classes. There are 3 cyclic subgroups of order 4, generated by a cycle of length 4 (each of these 3 groups has 2 generators). Further, there is the normal subgroup of $\mathfrak{S}_4$ which consists of the elements of order 2 which are product of disjoint transpositions:

$$V_4 = \{I, (a, b)(c, d),\ (a, c)(b, d),\ (a, d)(b, c)\}.$$

This is the abelian non cyclic Klein group of order 4. Furthermore, there are 3 non normal subgroups of order 4 isomorphic to V_4, generated by two disjoint transpositions

$$\{(a, b),\ (c, d),\ (a, b)(c, d)\}, \quad \{(a, c),\ (b, d),\ (a, c)(b, d)\}, \quad \{(a, d),\ (b, c),\ (a, d)(b, c)\}.$$

• 4 subgroups of order 6, and 1 conjugacy class: they are the subgroups isomorphic to $\mathfrak{S}_3$ which fix one of the 4 elements a, b, c, d and permute the 3 others.

• 3 subgroups of order 8, and 1 conjugacy class: they are isomorphic to the dihedral group D_4 of order 8

$$< (a, b), (a, c, b, d) >,\ < (a, c), (a, b, c, d) >,\ < (a, d), (a, b, d, c) > .$$

• 1 subgroup of order 12 and index 2, kernel of the signature: this is the alternating group $\mathfrak{A}_4$.

• 1 subgroup of order 24, namely $\mathfrak{S}_4$.

(c) The 4 normal subgroups are $\{1\}$, V, $\mathfrak{A}_4$ and $\mathfrak{S}_4$.

`Reference:`

`http://groupprops.subwiki.org/wiki/Subgroup_structure_of_symmetric_group:S4`

2. *The dihedral group D_n of order $2n$.*

(a) By induction, for $k \in \mathbb{Z}$ we have $r^k(i) = i + k \mod n$, hence $r^k = 1$ if and only if n divides k.

Also $s^2(i) = n + 2 - (n + 2 - i) = i \mod n$.

Further, $rs(i) = r(s(i)) = n + 3 - i \mod n$ and $srs(i) = s(rs(i)) = n + 2 - (n + 3 - i) = -1 + i \mod n$, hence $rsrs(i) = r(srs(i)) = i \mod n$.

It follows that

$$D_n = \{1, r, r^2, \ldots, r^{n-1}, s, sr, sr^2, \ldots, sr^{n-1}\} = \{1, r, r^2, \ldots, r^{n-1}, s, rs, r^2s, \ldots, r^{n-1}s\}.$$

For instance $D_1 = S_2 = C_2$, $D_2 = C_2 \times C_2$ (Klein group V_4), $D_3 = S_3$.

(b) Assume n is odd. Let a and b be generators of D_{2n} with a of order $2n$ and b of order 2 satisfying $abab = 1$. Since n is odd, the cyclic subgroup of D_{2n} of order $2n$ generated by a is the direct product of the cyclic group H of order 2 generated by a^n and the cyclic group generated by a^2.

Since $a^2b = ba^{-2}$, the subgroup K of D_{2n} generated by a^2 and b is isomorphic to D_n. We have

$$K = \{1, a^2, a^4 \dots, a^{2n-2}, b, ba^2, ba^4 \dots, ba^{2n-2}\}.$$

The elements a^i and a^ib belong to K if i is even, while if i is odd they can be written a^nc with $c \in K$. Hence $HK = D_{2n}$. Since a^n does not belong to K, we have $H \cap K = \{1\}$. Finally, since a^n has order 2, we have $a^nba^{-n} = b$, and a^n commutes with the elements of K. This proves that D_{2n} is the direct product of the two subgroups H, K, hence is isomorphic to the direct product $C_2 \times D_n$.

(c) Let G be a group of order $2p$ with p prime. Let us show that G is either the cyclic group of order $2p$ or else the dihedral group D_p. This is true for $p = 2$, since G is either the cyclic group of order 4 or else the Klein group V_4 which is the dihedral group D_2.

Assume p is odd. If G is commutative, it it the direct product of a group of order 2 and a group of order p, hence it is cyclic.

Assume G is not commutative. It contains a subgroup of order 2, say $H = \{1, s\}$, and a subgroup of order p, say K. The subgroup K of order p is normal, since its index is 2, and it is cyclic, any element of K other than 1 is a generator of K. Let r be a generator of K. We have

$$G = \{1, r, r^2, \dots, r^{n-1}, s, sr, sr^2, \dots, sr^{n-1}\}.$$

Since K is normal in G, and since $s^{-1} = s$, there exists $i \in \{0, \dots, p-1\}$ such that $srs = r^i$. The assumption that G is not commutative implies $i \neq 1$. From $s^2 = 1$ we deduce

$$r = s^2rs^2 = s(srs)s = sr^is = r^{i^2},$$

which implies that $i^2 \equiv 1 \mod p$. In the finite field $\mathbb{F}_p$ the only solution $\neq 1$ to the equation $x^2 = 1$ is $x = -1$. Hence $i = p - 1$, $srs = r^{-1}$ and D is the dihedral group D_p.

3. *A transitive subgroup of $\mathfrak{S}_n$ containing a $n-1$ cycle and a transposition is $\mathfrak{S}_n$.*

(a) Let $\sigma = (1, 2, \cdots, n-1)$ and $\tau = (1, n)$. We have

$$\sigma^{-1}(1) = n-1, \quad \tau(n-1) = n-1, \quad \sigma(n-1) = 1, \quad \text{hence} \quad \sigma\tau\sigma^{-1}(1) = 1.$$

For $3 \le i \le n-1$ we have $2 \le i-1 \le n-2)$, hence

$$\sigma^{-1}(i) = i-1, \quad \tau(i-1) = i-1, \quad \sigma(i-1) = i, \quad \text{hence} \quad \sigma\tau\sigma^{-1}(i) = i.$$

Also we have

$$\sigma^{-1}(2) = 1, \quad \tau(1) = n, \quad \sigma(n) = n, \quad \text{hence} \quad \sigma\tau\sigma^{-1}(2) = n$$

and

$$\sigma^{-1}(n) = n, \quad \tau(n) = 1, \quad \sigma(1) = 2, \quad \text{hence} \quad \sigma\tau\sigma^{-1}(n) = 2.$$

Therefore

$$\sigma\tau\sigma^{-1} = (2, n).$$

(b) We deduce from (a) that for $0 \le k \le n-2$,

$$\sigma^k\tau\sigma^{-k} = (k+1, n).$$

Since $\mathfrak{S}_n$ is generated by the $n-1$ transpositions $(1, n), (2, n), \dots, (n-1, n)$, it is also generated by τ and σ.

(c) Let G be a transitive subgroup of $\mathfrak{S}_n$ containing a $n-1$ cycle and a transposition. We label the elements in such a way that the cycle of length $n-1$ is $(1, 2, \dots, n-1)$. Since G is transitive, the transposition does not fix n. We permute the elements, if necessary, so that the transposition is $(1, n)$. From (b) we deduce that $G = \mathfrak{S}_n$.