MODULE 3: EXERCISE SHEET 2

These problems are due Sunday, 19 June, 2016. They must be sent to nap@rnta.eu (copy to nickgill@cantab.net) by 10 pm Nepal time.

(1) Prove that the extension $\mathbb{Q}[\sqrt{2}+\sqrt{3}]/\mathbb{Q}$ is Galois, and that the associated Galois group is non-cyclic.[1]

(2) Prove Lemma 1.35 of Milne. (i.e. Show that every number in $\mathbb{Q}$ is constructible.)

(3) (a) Suppose that G is a group of order 2^k for some integer $k \geq 1$. Prove that $Z(G)$, the center of G, is non-trivial.

(b) Use (a) to prove that if G is a group of order 2^k for some integer $k \geq 1$, then there is a series of groups

$$\{1\} = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_k = G$$

such that $|G_i : G_{i-1}| = 2$ for $i = 1, \dots, k$.

(c) Use (b) to prove that if α is a real number that lies in a Galois extension $E/\mathbb{Q}$ with $|\operatorname{Gal}(E/\mathbb{Q})| = 2^k$ for some integer k, then α is constructible.

(4) Construct a regular pentagon. (This is tricky: you might wish to use wikipedia, in which case please make sure that you explain your answer carefully.)

(5) Suppose that $P = (x, y)$ is a constructible point. Let α and β be elements of $\mathbb{Q}(x, y)$. Show that (α, β) is a constructible point.

(6) Prove that if a regular p-gon can be constructed, then a regular $2p$-gon can be constructed.

(7) Can a regular 9-gon be constructed?

(8) Do Exercise 1.4 of Milne. (i.e. Prove that with straight-edge, compass, and angle trisector, it is possible to construct a regular 7-gon – refer to E.g. 3.21).

(9) **(Revision)** Show that over $\mathbb{F}_2$, one can find polynomials of degree ≤ 3 with Galois group $\{1\}$, C_2 and C_3 but **not** S_3.

(10) **(Optional extra)** Do the previous question for $\mathbb{F}_3$ instead of $\mathbb{F}_2$.

(11) **(Optional extra)** Prove that the origami-construction of $\sqrt[3]{2}$ that was done in class is correct.

[1]This exercise corrects an erroneous assertion that I made verbally in the first week of classes.