2016 NAP Lecture Part II, Problem 4

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Restatement of Remark 3.11 (a) Let E/F be Galois with $G = \mathrm{Gal}(E/F)$. Take an element $\alpha \in E$, and let $\alpha_1 = \alpha, \alpha_2, \cdots, \alpha_m$ be the orbit of α under the action of G. They are called the conjugates of α. In the argument of the proof of the above " (b) $\Longrightarrow$ (c)" we showed that $f(X) = \prod_i (X - \alpha_i)$ is the minimum polynomial of α.

(b) Let E/F be a finite extension, and let G be a finite subgroup of $\mathrm{Aut}(E/F)$. Then Prop. 2.7 (a) says $\sharp\mathrm{Aut}(E/E^G) = [E : E^G]$. On the other hand, by (3.5), it holds $(\mathrm{Gal}(E/E^G)) = G = \mathrm{Aut}(E/E^G)$. So we have $\sharp G = [E : E^G]$.

(b-bis) Especially, in case E/F is Galois, we have

(i) $\mathrm{Gal}(E/F) = \mathrm{Aut}(E/F)$, (ii) $\sharp\mathrm{Gal}(E/F) = [E : F]$.

Problem II-12] = Exercise 2-2

(a) Set $\mathrm{char}(F) = p(> 0)$.Show that if $X^p - X - a$ is reducible in $F[X]$, it has only simple factors. (hint. see Prop. 2.12)

(b) Show that $X^p - X - 1$ is irreducible in $\boldsymbol{F}_p[X]$. (hint. use (a), but not so easy)

(c) Show that $X^p - X - 1$ is irreducible in $\boldsymbol{Q}[X]$. (hint. use (b))

Problem II-13] = Exercise 2-3

(a)Find the splitting field E_f of $f(X) = X^5 - 2 \in \boldsymbol{Q}[X]$.

(b) Determine $[E_f : \boldsymbol{Q}]$.

Problem II-14] = Exercise 2-4

Set $f(X) = X^{p^m} - 1 \in \boldsymbol{F}_p[X]$. (i) Find the splitting field E_f of f. (ii) Determine $[E_f : \boldsymbol{F}_p]$.

NAP Lecture Part II Exercises Sheet

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