NAP PROBLEM SET #2

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These problems are due Sunday, 22 May, at 10 pm.

IMPORTANT: Send your solutions to nap@rnta.eu with a copy to rwiegand1@unl.edu and swiegand1@unl.edu by 10 pm Nepal time.

1. Let F be a field, and let $f(X) \in F[X]$ have degree d. Prove that $f(X)$ has at most d distinct roots in F. Find an example of a commutative ring R and a polynomial $f(X) \in R[X]$ of degree 2, such that $f(X)$ has 32 distinct roots in R.

2. Let $F \subseteq K$ be a field extension, and let $\alpha \in K$ be transcendental over F.
(a) Prove that $F[\alpha]$ and $F[X]$ are isomorphic rings.
(b) Prove that $F(\alpha)$ and $F(X)$ are isomorphic fields.

3. Let $F \subseteq K$ be a field extension. Let α be an element of K whose degree over F is odd. Prove that $F(\alpha) = F(\alpha^2)$.

4. Let $F \subseteq K$ be a field extension, and let α and β be elements of K whose degrees over F are m and n, respectively. Prove that if $\gcd(m, n) = 1$, then $[F(\alpha, \beta)] = mn$.

5. Let $F \subseteq K$ be a field extension, and let α and β be elements of K whose degrees over F are m and n, respectively. Prove that α has degree m over $F(\beta)$ if and only if β has degree n over $F(\alpha)$.

6. Let $F \subseteq K$ be a field extension, let α be an element of K and let $p(X)$ be the minimal polynomial of α over F. Let $f(X) \in F[X]$ be a monic polynomial of degree d. Prove that
(a) $f(\alpha) = 0 \iff f(X) \in (p(X))$.
(b) If $\beta \in K$ is such that $f(\beta) = 0$, then $d = [F(\alpha) : F] \iff f(X)$ is the minimal polynomial of β over F.

7. Let $\alpha = \sqrt{3} + \sqrt{5} \in \mathbb{C}$.
(a) Find the minimal polynomial of α over $\mathbb{Q}$, and *prove* that it is the minimal polynomial.
(b) Prove that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{3}, \sqrt{5})$.

You may discuss problems with other students in the class, but you must do the write-up completely by yourself, without consulting anyone else. You may refer, by number, to theorems, propositions, etc., in Milne's book, provided that they are results that we have covered in the the first two weeks of the class. If you are ambitious, you can write your solutions in TeX and send them as an attachement. Alternatively, you can write them out (legibly, please), scan them, and send as an attachment. A less desirable option would be to photograph your solutions and send the photo; this will probably be harder to read, so the first two options are preferable.

You can download Milne's book at:

http://www.jmilne.org/math/CourseNotes/FT.pdf

The NAP website is: http://www.rnta.eu/nap/

Feel free to email us anytime with questions about the course or the homework, or about other mathematical issues!

Date: 16 May, 2016.